

1

Liebmann technical documentation



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Laplace equation 2D (ZR)

4

(Cylindrical coordinates).

5

relaxation scheme explained.

6

(5 - point star)

7

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14	Contents	
15	1 Liebmann technical documentation series	4
16	2 Versions of this document	4
17	3 Solving Laplace equation using relaxation method	5
18	4 Explanation of symbols in calculations	6
19	5 Mesh ZR - type A (on axis)	7
20	6 Mesh ZR - type B (on axis)	8
21	7 Mesh ZR - type C (on axis)	9
22	8 Mesh ZR - type D (on axis)	10
23	9 Example of A-type mesh in ANSI C (on axis)	11
24	10 Example of B-type mesh in ANSI C (on axis)	14
25	11 Example of C-type mesh in ANSI C (on axis)	15
26	12 Example of D-type mesh in ANSI C (on axis)	17
27	13 Partial derivatives on Oz axis	18
28	13.1 Personal note	18
29	13.2 Nodes numbering (on axis Oz)	18
30	13.3 Taylor expansion of $V_{(z,r)}$ function in cylindrical coordinates . . .	18
31	13.4 Laplace operator in rotationally symmetrical systems	18
32	13.5 Value of first partial derivative of V with respect to r on axis Oz .	19
33	13.6 Value of second partial derivative of V with respect to r on axis Oz	19
34	13.7 Value of first partial derivative of V with respect to r divided by r	
35	on axis Oz	20
36	13.8 Value of second partial derivative of V with respect to z on axis Oz	21
37	14 Partial derivatives off Oz axis	22
38	14.1 Personal note	22
39	14.2 Nodes numbering in Liebmann mesh (off axis Oz)	22
40	14.3 Taylor expansion of $V_{(z,r)}$ function in cylindrical coordinates . . .	22
41	14.4 Laplace operator in rotationally symmetrical systems	23
42	14.5 Value of second partial derivative of V with respect to r off axis Oz	23
43	14.6 Value of first partial derivative of V with respect to r divided by r	
44	off axis Oz	23
45	14.7 Value of second partial derivative of V with respect to z off axis Oz	23

46	15 Relaxation formula for node P1 (on axis Oz)	24
47	15.1 Node description	24
48	15.2 Calculation of relaxation formula	24
49	15.3 Final forms of relaxation formula	25
50	15.3.1 zrLV_RELAX5_P1_ON_A	25
51	15.3.2 zrLV_RELAX5_P1_ON_B	25
52	15.3.3 zrLV_RELAX5_P1_ON_C	25
53	15.3.4 zrLV_RELAX5_P1_ON_D	25
54	16 Relaxation formula for node P2 (on axis Oz)	26
55	16.1 Node description	26
56	16.2 Calculation of relaxation formula	26
57	16.3 Final forms of relaxation formula	27
58	16.3.1 zrLV_RELAX5_P2_ON_A	27
59	16.3.2 zrLV_RELAX5_P2_ON_B	27
60	16.3.3 zrLV_RELAX5_P2_ON_C	27
61	16.3.4 zrLV_RELAX5_P2_ON_D	27
62	17 Relaxation formula for node P3 (on axis Oz)	28
63	17.1 Node description	28
64	17.2 Calculation of relaxation formula	28
65	17.3 Final forms of relaxation formula	29
66	17.3.1 zrLV_RELAX5_P3_ON_A	29
67	17.3.2 zrLV_RELAX5_P3_ON_B	29
68	17.3.3 zrLV_RELAX5_P3_ON_C	29
69	17.3.4 zrLV_RELAX5_P3_ON_D	29
70	18 Relaxation formula for node P4	30
71	18.1 Node description	30
72	18.2 Calculation of relaxation formula	30
73	18.3 Final forms of relaxation formula	31
74	18.3.1 zrLV_RELAX5_P4_A	31
75	18.3.2 zrLV_RELAX5_P4_B	31
76	18.3.3 zrLV_RELAX5_P4_C	31
77	18.3.4 zrLV_RELAX5_P4_D	31
78	19 Relaxation formula for node P5	32
79	19.1 Node description	32
80	19.2 Calculation of relaxation formula	32
81	19.3 Final forms of relaxation formula	33
82	19.3.1 zrLV_RELAX5_P5_A	33
83	19.3.2 zrLV_RELAX5_P5_B	33
84	19.3.3 zrLV_RELAX5_P5_C	33
85	19.3.4 zrLV_RELAX5_P5_D	33

86	20 Relaxation formula for node P6	34
87	20.1 Node description	34
88	20.2 Calculation of relaxation formula	34
89	20.3 Final forms of relaxation formula	35
90	20.3.1 zrLV_RELAX5_P6_A	35
91	20.3.2 zrLV_RELAX5_P6_B	35
92	20.3.3 zrLV_RELAX5_P6_C	35
93	20.3.4 zrLV_RELAX5_P6_D	35
94	21 Relaxation formula for node P7	36
95	21.1 Node description	36
96	21.2 Calculation of relaxation formula	36
97	21.3 Final forms of relaxation formula	37
98	21.3.1 zrLV_RELAX5_P7_A	37
99	21.3.2 zrLV_RELAX5_P7_B	37
100	21.3.3 zrLV_RELAX5_P7_C	38
101	21.3.4 zrLV_RELAX5_P7_D	38
102	22 Relaxation formula for node P8	39
103	22.1 Node description	39
104	22.2 Calculation of relaxation formula	39
105	22.3 Final forms of relaxation formula	40
106	22.3.1 zrLV_RELAX5_P8_A	40
107	22.3.2 zrLV_RELAX5_P8_B	40
108	22.3.3 zrLV_RELAX5_P8_C	40
109	22.3.4 zrLV_RELAX5_P8_D	41
110	23 Relaxation formula for node P9	42
111	23.1 Node description	42
112	23.2 Calculation of relaxation formula	42
113	23.3 Final forms of relaxation formula	43
114	23.3.1 zrLV_RELAX5_P9_A	43
115	23.3.2 zrLV_RELAX5_P9_B	43
116	23.3.3 zrLV_RELAX5_P9_C	44
117	23.3.4 zrLV_RELAX5_P9_D	44

118 **1 Liebmann technical documentation series**

- 119 1. Wyznaczanie rozkładu pola elektrostatycznego w próżni metodą relak-
120 sacyjną Liebmann. (Polish version / wersja polska)
- 121 2. Determination of electrostatic field distribution by using Liebmann relax-
122 ation method. (English version / wersja angielska)
- 123 3. Graphics. Mapping voltages to colours. (colormaps)
- 124 4. Laplace equation 2D (XY). (Cartesian coordinates). Relaxation scheme
125 explained. (5 - point star)
- 126 5. Laplace equation 2D (ZR) (Cylindrical coordinates). Relaxation scheme
127 explained. (5 - point star)
- 128 6. Liebmann source code. (ANSI C programming language)

129 **2 Versions of this document**

- 130 1. version 1 - 2023.11.03
- 131 2. version 2 - 2023.01.04
- 132 3. version 3 - 2024.02.02
- 133 4. version 4 - 2024.04.02
- 134 5. version 5 - 2024.05.18
- 135 6. version 6 - 2024.05.23
- 136 7. version 7 - 2024.05.24
- 137 8. version 8 - 2024.06.06 (complete $P_1..P_9$)
- 138 9. version 9 - 2024.06.09
- 139 10. version 10 - 2024.07.17
- 140 11. version 11 - 2024.07.18
- 141 12. version 12 - 2024.09.03

142 **3 Solving Laplace equation using relaxation method**

143 I tried to solve Laplace equation using mainly information from Pierre Grivet's
144 book (Electron Optics) - [1].

145 There are few editions of this book (1965, 1972). Second edition (1972) con-
146 tains explanation of relaxation method (page 38).

147 More generalized approaches has been drafted by James R. Nagel - [2].
148 <https://my.ece.utah.edu/~ece6340/LECTURES/Feb1/> (visited 2023-03-01).

149
150 Taylor expansion in cylindrical coordinates has been found on the Internet:
151 [3].

152
153 There are also publications edited by Albert Septier: Focusing of Charged
154 Particles [4] and Applied Charged Particle Optics (part A). [5].

155 I have also found some ideas in publication of D W O Heddle: Electrostatic
156 Lens Systems [6] (especially using PC computers to solve electrostatic prob-
157 lems).

158 I have also found (brief) description of by - hand solving of Laplace equa-
159 tion by Bohdan Paszkowski - [7] (Polish edition). English translation of this book
160 also exists - [8].

161
162 I would like to thank many people, who helped me with this challenge. Espe-
163 cially prof. dr hab. Mieczysław Jałochowski (supervisor of my master's thesis),
164 who enabled me to use SIMION and MATLAB software while writing master's
165 thesis about electron optical systems at University of Maria Curie - Skłodowska
166 in Lublin in 2008. I would also thank to prof. Marcin Turek for fruitful discus-
167 sion about numerical methods. What is more, my colleague Bartosz in 2012
168 had explained me general problems with software efficiency. So he had also
169 contributed significantly to the idea of Liebmann software (especially using C
170 language).

171 **4 Explanation of symbols in calculations**

- 172 • P_i - i -th mesh node
- 173 • V_i - value of electrostatic potential at node P_i . Unit - [V]
- 174 • h - mesh step (for example h_x - mesh step in x direction). Unit - [mm]
- 175 • $g_{i+/-}$ - gradient in direction i (for example $g_{1z-} = \frac{V_1 - V_{1z-}}{h_z}$). Unit - [$\frac{V}{mm}$]
- 176 • i_{row} - index of row in mesh. Values of $i_{row} = 1, 2, \dots, \text{size_row}$
- 177 • i_{col} - index of column in mesh. Values of $i_{col} = 1, 2, \dots, \text{size_col}$
- 178 • p - in book: - [1] $r = ph_r$, so for off - axis point we have: $p = (i_{row} - 1)$

179 Symbols in final relaxation formulae

180 zrLV_RELAX5_P1_A

- 181 • zr - coordinates (2D, cylindrical)
- 182 • LV - Laplace equation in vacuum (no dielectrics)
- 183 • RELAX_5 - 5- point relaxation method
- 184 • P1 - relaxation scheme for point P1 (in general P1 .. P9)
- 185 • A - mesh type A (in general A .. D)

186 **5 Mesh ZR - type A (on axis)**

187 $h_z \neq h_r$

188 gradient V outside a mesh exists

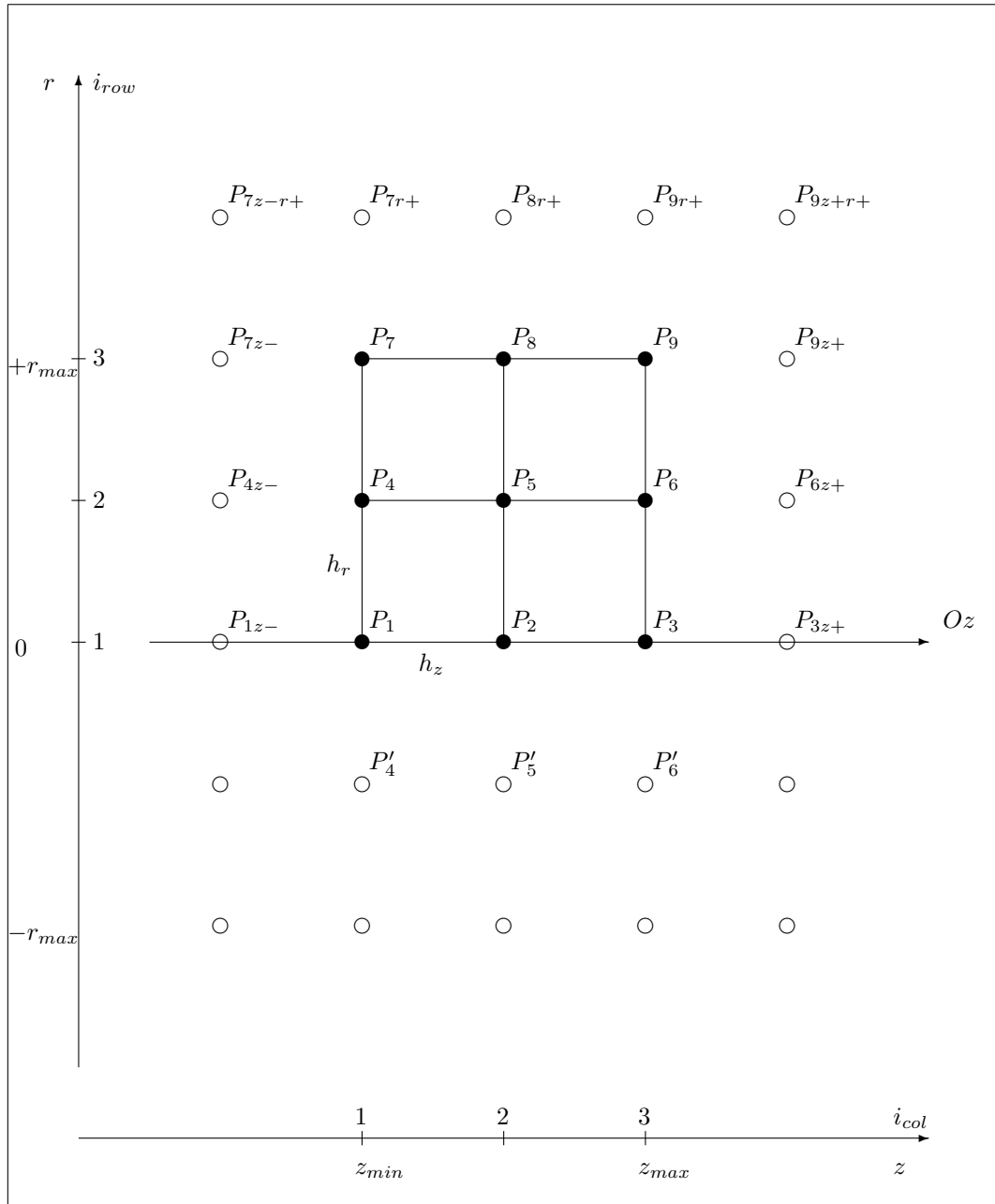


Figure 1: Mesh ZR type A

189 **6 Mesh ZR - type B (on axis)**

190 $h_z \neq h_r$

191 gradient V outside a mesh does not exist

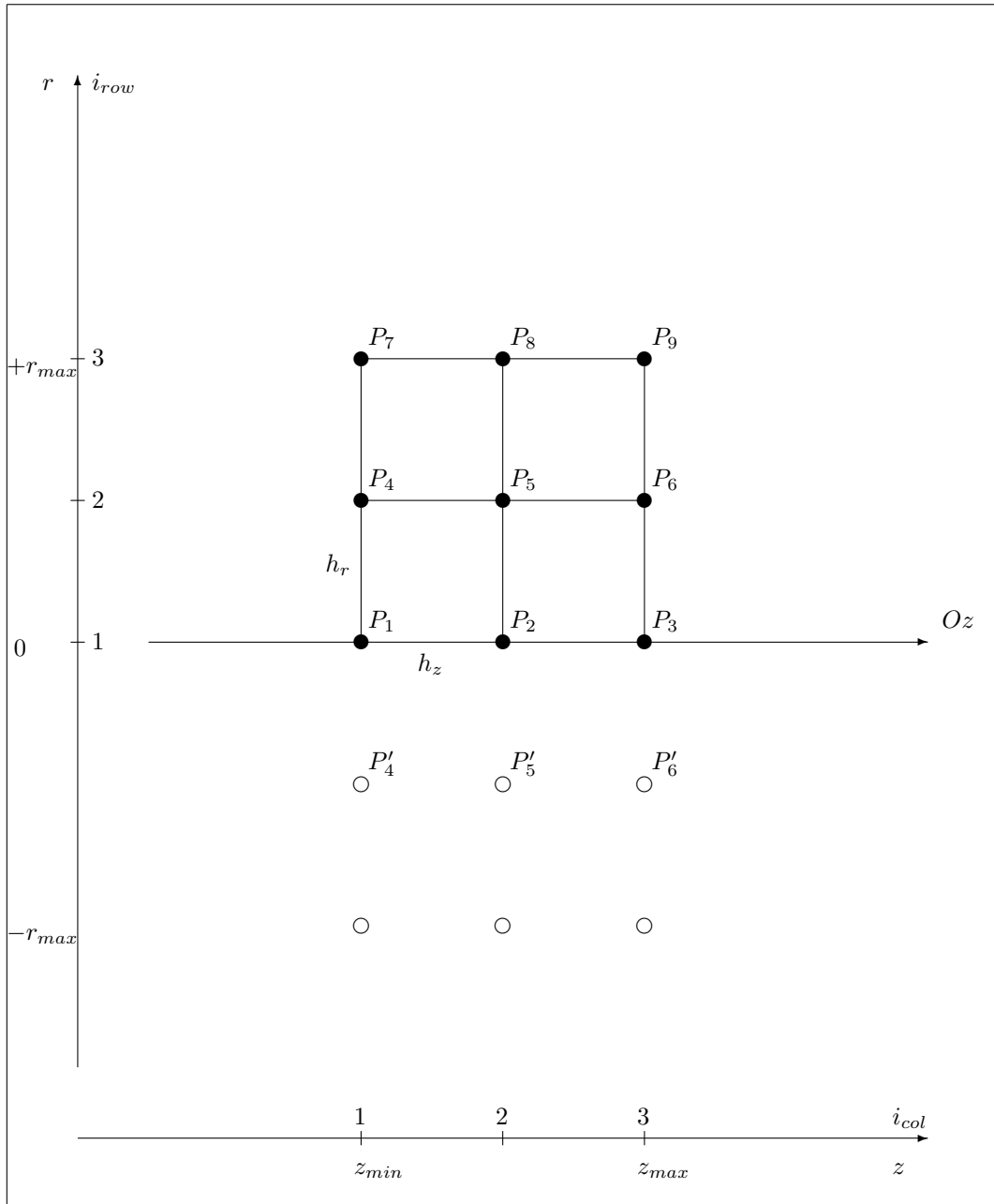


Figure 2: Mesh ZR type B

192 **7 Mesh ZR - type C (on axis)**

193 $h_z = h_r = h$

194 gradient V outside a mesh exists

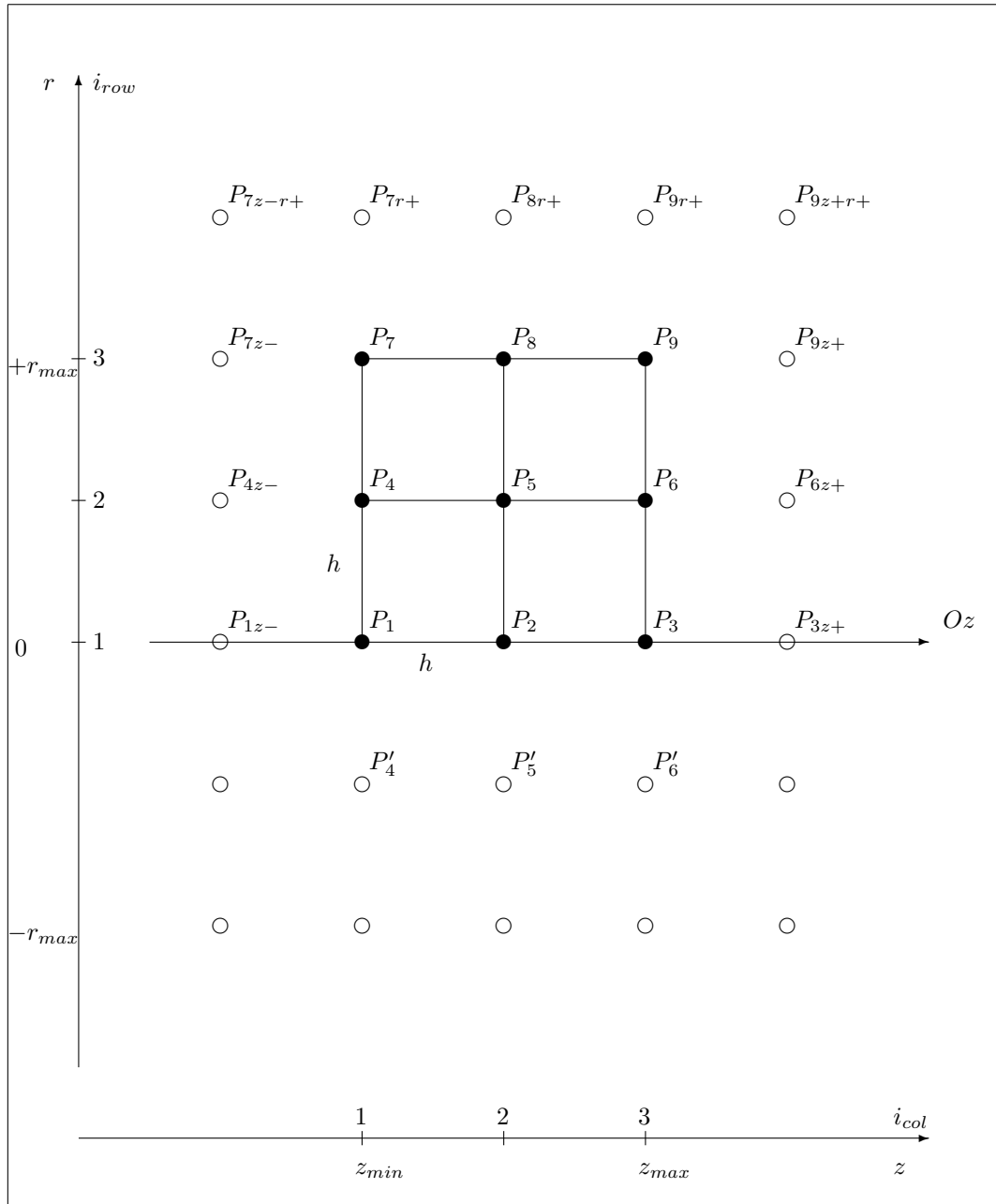


Figure 3: Mesh ZR type C

195 **8 Mesh ZR - type D (on axis)**

196 $h_z = h_r = h$

197 gradient V outside a mesh does not exist

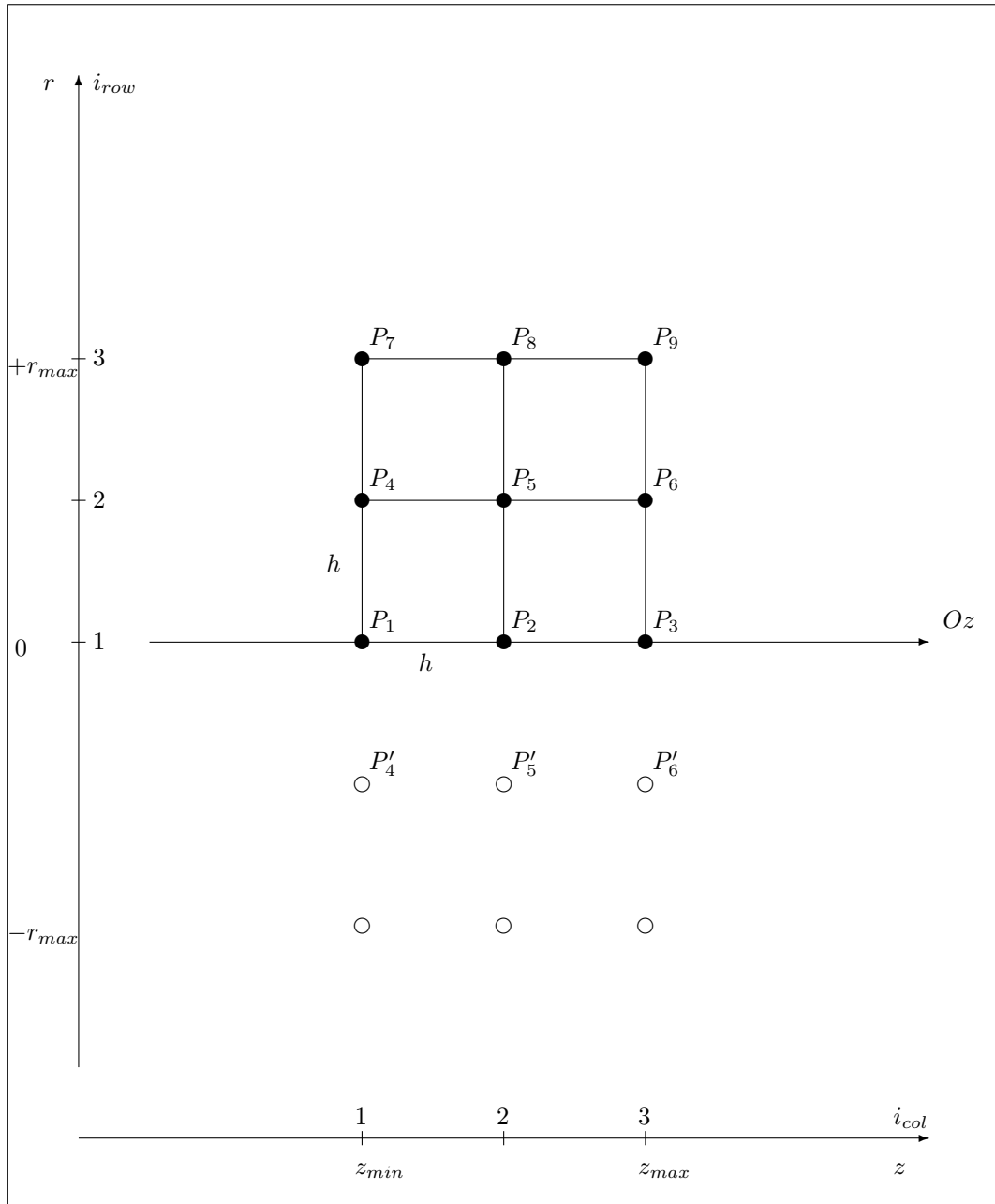


Figure 4: Mesh ZR type D

198 **9 Example of A-type mesh in ANSI C (on axis)**

199 Example of A- type mesh in ANSI C program. The mesh is represented by 2
 200 dimensional array of double precision numbers. Rows and columns in mesh
 201 are numbered from 1 (this was my choice) instead of default 0 (as usual in C
 202 language). This choice has pros and cons. It is easier to calculate mesh size
 203 (size_row * size_col). Access to each node can be also more intuitive, but logic
 204 in each library function must contain this shift between node ordering styles.

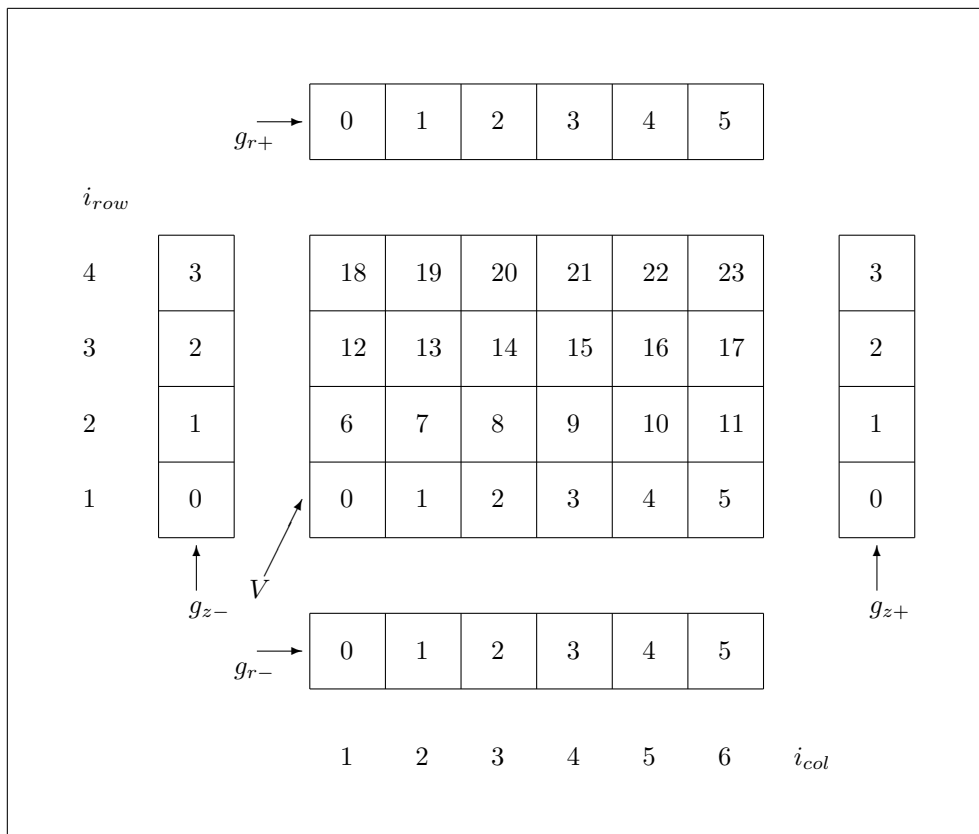


Figure 5: ANSI C - mesh XY type A

205 Note. This is more general example of „off-axis” mesh. If bottom edge of
 206 mesh lies on axis Oz , then gradient g_{r-} does not exist.

- 207 • $g_{z-} \equiv \text{double* ptr_gZ_minus}$
- 208 • $g_{z+} \equiv \text{double* ptr_gZ_plus}$
- 209 • $g_{r-} \equiv \text{double* ptr_gR_minus}$
- 210 • $g_{r+} \equiv \text{double* ptr_gR_plus}$

```

211 •  $V \equiv \text{double}^* \text{ptr}_V$ 
212 • unsigned int size_row == 4
213 • unsigned int size_col == 6
214 • unsigned int i_row == 1, 2, .., 4
215 • unsigned int i_col == 1, 2, .., 6
216 • double h_z == 1.0 [mm]
217 • double h_r == 2.0 [mm]

```

218 The following picture describes analogous version of `ptr_V` mesh, which
 219 can be dynamically allocated on heap by pointer method. The mesh is represented by
 220 single block of memory. The numbers of rows and columns are also known, so each node
 221 can be also accessed by appropriate index (memory address).
 222

223 The following picture describes analogous version of mesh, which can be
 224 easily dynamically allocated on heap by pointer method. The mesh is represented by
 225 single block of memory. The numbers of rows and columns are also known, so each node
 226 can be also accessed by appropriate index (memory address).
 227

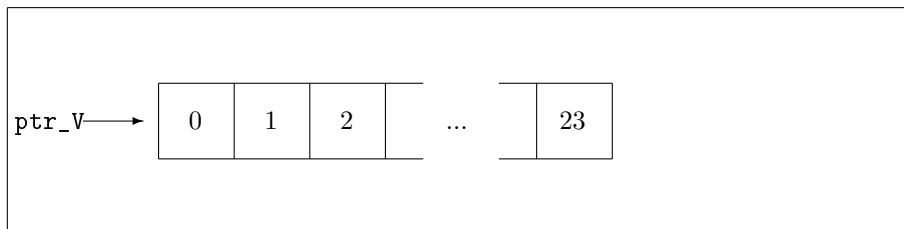


Figure 6: ANSI C - mesh ZR type D

228 Each mesh point has its unique index (let's say `icp` - (index of central
 229 point)), which can be determined, if we know indices of row and column (`i_row`,
 230 `i_col`).

$$\text{icp} == (\text{i_row} - 1) * \text{size_col} + \text{i_col} - 1 \quad (9.1)$$

231 For example for each point of a mesh indices of row and column have values:
 232

$$\begin{aligned} \text{i_row} &== 1, 2, \dots, \text{size_row} \\ \text{i_col} &== 1, 2, \dots, \text{size_col} \end{aligned} \quad (9.2)$$

233 Also for any relaxation formula for off - axis case the p symbol appears. This
234 symbol is connected with r cylindrical coordinate of given node:
235

$$r = ph_r \quad (9.3)$$

236 SO:

$$p == (i_row - 1) \quad (9.4)$$

237 **10 Example of B-type mesh in ANSI C (on axis)**

238 Example of B- type mesh in ANSI C program. The mesh is analogous to A -
 239 type mesh. There are no electric field gradients on mesh borders.

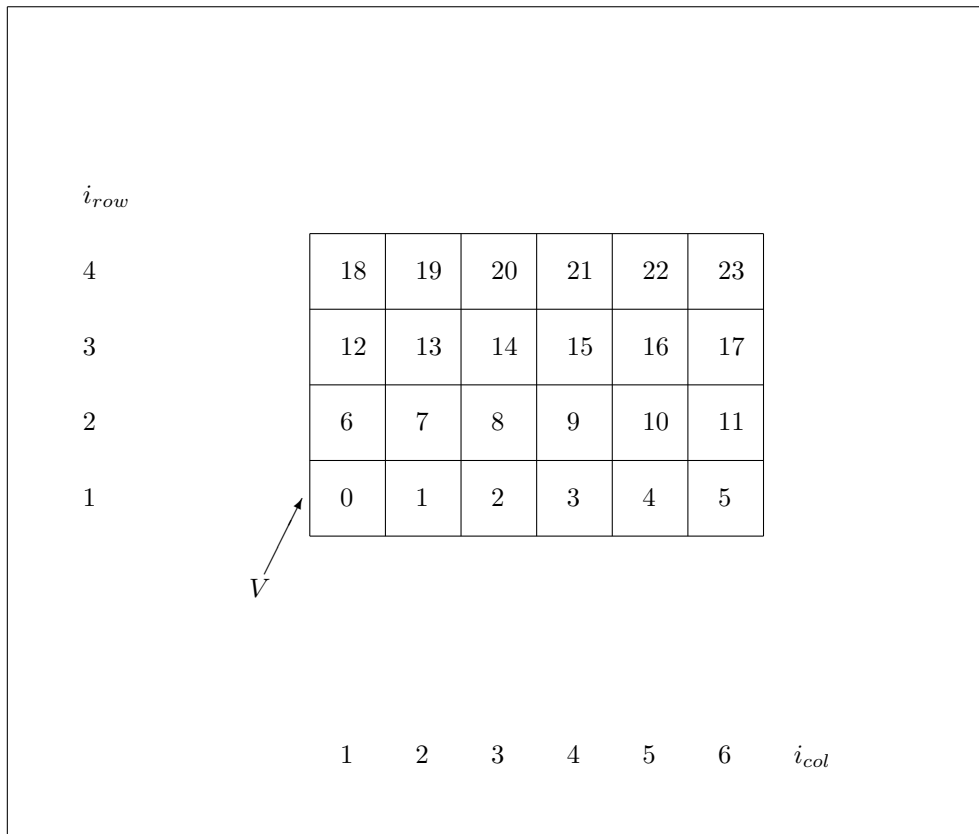


Figure 7: ANSI C - mesh XY type B

- 240 • $V \equiv \text{double* ptr}_V$
- 241 • unsigned int size_row == 4
- 242 • unsigned int size_col == 6
- 243 • unsigned int i_row == 1, 2, .., 4
- 244 • unsigned int i_col == 1,2, .., 6
- 245 • double h_z == 1.0 [mm]
- 246 • double h_r == 2.0 [mm]

247 **11 Example of C-type mesh in ANSI C (on axis)**

248 Example of C- type mesh in ANSI C program. The mesh is analogous to A -
 249 type mesh. Just mesh mesh step $h_x = h_y = h$.

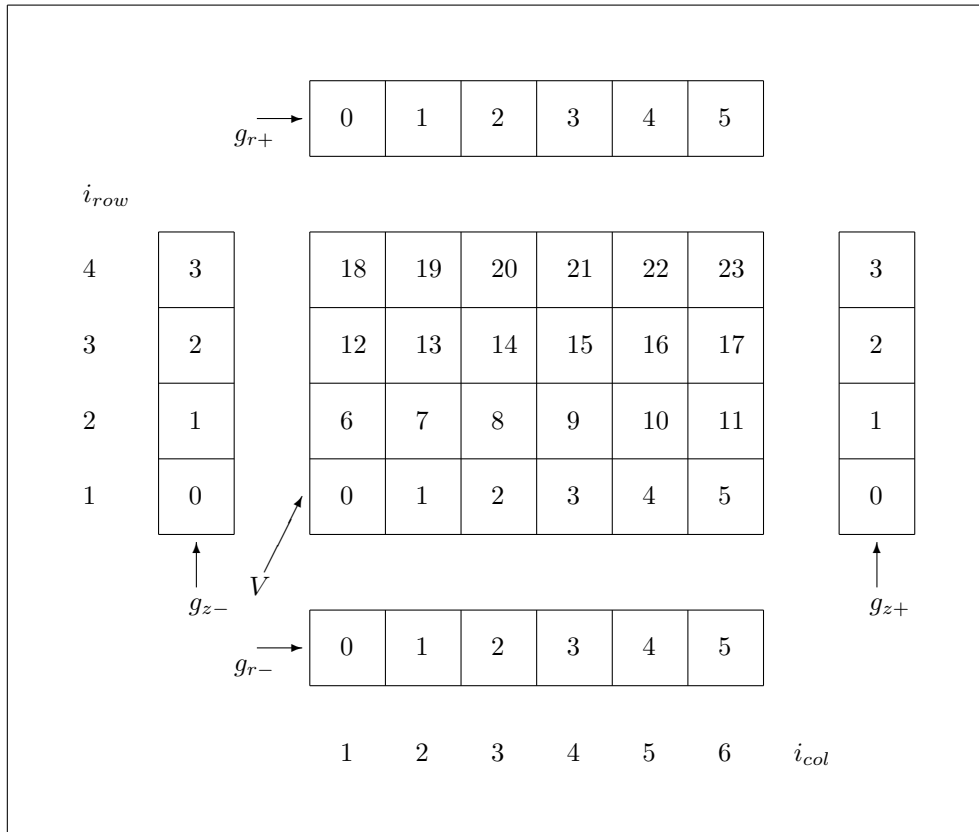


Figure 8: ANSI C - mesh XY type C

250 Note. This is more general example of „off-axis” mesh. If bottom egde of
 251 mesh lies on axis Oz , then gradient g_{r-} does not exist.

- 252 • $g_{z-} \equiv \text{double* ptr_gZ_minus}$
- 253 • $g_{z+} \equiv \text{double* ptr_gZ_plus}$
- 254 • $g_{r-} \equiv \text{double* ptr_gR_minus}$
- 255 • $g_{r+} \equiv \text{double* ptr_gR_plus}$
- 256 • $V \equiv \text{double* ptr_V}$
- 257 • `unsigned int size_row == 4`


```
258 • unsigned int size_col == 6
259 • unsigned int i_row == 1, 2, .., 4
260 • unsigned int i_col == 1,2, .., 6
261 • double h == 1.0 [mm]
```

262 **12 Example of D-type mesh in ANSI C (on axis)**

263 Example of D- type mesh in ANSI C program. The mesh is analogous to B -
264 type mesh. Just $h_x = h_y = h$.

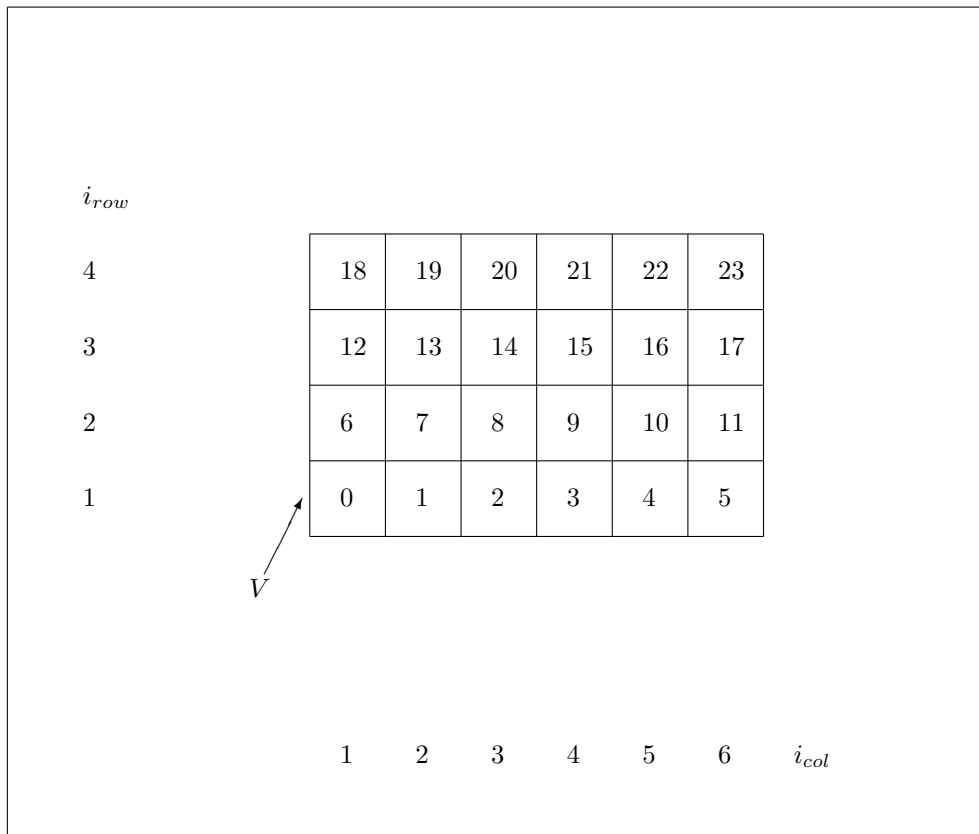


Figure 9: ANSI C - mesh ZR type D

- 265 • $V \equiv \text{double* ptr}_V$
- 266 • `unsigned int size_row == 4`
- 267 • `unsigned int size_col == 6`
- 268 • `unsigned int i_row == 1, 2, .., 4`
- 269 • `unsigned int i_col == 1,2, .., 6`
- 270 • `double h == 1.0 [mm]`

271 **13 Partial derivatives on Oz axis**

272 **13.1 Personal note**

273 This is my personal interpretation. I cannot guarantee correctness of this ap-
274 proach

275 **13.2 Nodes numbering (on axis Oz)**

276 We will try to work with P_2 point (determine approximations of aprtial derivatives
277 for point P_2 , which lies on axis Oz). Nodes numbering on axis Oz differs from
278 numbering convention in Pierre Grivet's book.

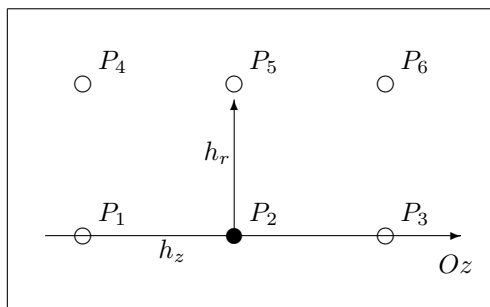


Figure 10: Nodes on axis Oz

279 Point P_2 is situated on Oz axis. It has 2 neighbours on axis Oz : P_1 and P_3 .
280 Node P_5 lies above P_2 node. The mesh step in r direction is h_r . The mesh
281 step in z direction is h_z .

282 **13.3 Taylor expansion of $V_{(z,r)}$ function in cylindrical coordinates**

283 Taylor expansion of $V_{(z,r)}$ function in cylindrical coordinates [3]:

$$\begin{aligned}
 V_{(z,r)} = V_{(z_0,r_0)} + \left(\frac{\partial V}{\partial z} \right)_{(z_0,r_0)} (z - z_0) + \\
 \left(\frac{\partial V}{\partial r} \right)_{(z_0,r_0)} (r - r_0) + \\
 \frac{1}{2!} \left(\frac{\partial^2 V}{\partial z^2} \right)_{(z_0,r_0)} (z - z_0)^2 + \dots
 \end{aligned} \tag{13.1}$$

284 **13.4 Laplace operator in rotationally symmetrical systems**

285 Laplace operator in cylindrical coordinates (cylindrical symmetry) consist of 3
286 elements [1] (on page 42):

$$\nabla^2 (V_{(z,r)}) = \left(\frac{\partial^2 V}{\partial r^2} \right) + \frac{1}{r} \left(\frac{\partial V}{\partial r} \right) + \left(\frac{\partial^2 V}{\partial z^2} \right) \quad (13.2)$$

287 In this chapter we will try to determine approximation of each term.

288 **13.5 Value of first partial derivative of V with respect to r on axis**
 289 Oz

290 In cylindrically symmetrical field first partial derivative of V (by r) on axis Oz
 291 equals zero (because $V_{(+dr)} = V_{(-dr)}$)

$$\left(\frac{\partial V}{\partial r} \right)_{(z,r=0)} = 0 \quad (13.3)$$

292 **13.6 Value of second partial derivative of V with respect to r on**
 293 **axis Oz**

294 In this subchapter we will try to determine the first term of equation 13.2

295 In our case there is node P_2 on axis Oz . The nearest neighbour of P_2 is
 296 node P_5 , which lies „over Oz axis“. The distance between P_2 and P_5 is h_r .
 297 When we „walk away“ axis Oz in r direction (from point P_2 to point P_5), the
 298 electric potential V_5 can be determined from truncated Taylor expansion 13.1
 299 by expression:

$$V_5 \approx V_2 + \left(\frac{\partial V}{\partial r} \right)_{P_2} \cdot h_r + \frac{1}{2!} \left(\frac{\partial^2 V}{\partial r^2} \right)_{P_2} \cdot h_r^2 \quad (13.4)$$

300 We want to determine the second derivative:

$$\left(\frac{\partial^2 V}{\partial r^2} \right)_{P_2} = ? \quad (13.5)$$

301 We solve equation 13.4 (using relation 13.3).

$$\left(\frac{\partial^2 V}{\partial r^2} \right)_{P_2} \approx \frac{2!(V_5 - V_2)}{h_r^2} = \frac{2(V_5 - V_2)}{h_r^2} \quad (13.6)$$

302 This is final form of approximation of the second derivative of V with respect
 303 to r on axis Oz . It will help us to determine Laplace operator in rotationally
 304 symmetrical systems.

305 **13.7 Value of first partial derivative of V with respect to r divided**
 306 **by r on axis Oz**

307 We will try to determine the second term of relation 13.2 When we are on Oz
 308 axis, the second term has to be determined (because it aims to value $\frac{0}{0}$).

309 When we „ walk away” axis Oz in r direction, the electric potential $V_{(z_0,r)}$
 310 can be determined from truncated Taylor expansion by:

311

$$V_{(z_0,r)} \approx V_{(z_0,r_0)} + \left(\frac{\partial V}{\partial r} \right)_{(z_0,r_0)} (r - r_0) \quad (13.7)$$

312 On Oz axis $r_0 = 0$, so $(r_0 - r) = r$

313

314 Thus we have:

$$V_{(z_0,r)} \approx V_{(z_0,0)} + \left(\frac{\partial V}{\partial r} \right)_{(z_0,0)} \cdot r \quad (13.8)$$

315 Now let us differentiate (both sides) of such relation:

$$\left| \frac{\partial}{\partial r} \right. \quad (13.9)$$

316 We get:

$$\left(\frac{\partial V}{\partial r} \right)_{(z_0,r)} \approx \left(\frac{\partial V}{\partial r} \right)_{(z_0,0)} + \left(\frac{\partial^2 V}{\partial r^2} \right)_{(z_0,0)} \cdot r + \left(\frac{\partial V}{\partial r} \right)_{(z_0,0)} \cdot 1 \quad (13.10)$$

317 On axis Oz we can apply relation 13.3. That's why we can remove these
 318 two terms (first and third) from equation 13.10:

319 So we get (if $r = 0$):

$$\left(\frac{\partial V}{\partial r} \right)_{(z_0,r)} \approx \left(\frac{\partial^2 V}{\partial r^2} \right)_{(z_0,0)} \cdot r \quad (13.11)$$

320 We can now divide both sides by r .

$$\left| \cdot \frac{1}{r} \right. \quad (13.12)$$

321 We have relation, which has been published in Pierre Grivet's book[1].

$$\left(\frac{1}{r} \frac{\partial V}{\partial r} \right)_{(z_0,r)} \approx \left(\frac{\partial^2 V}{\partial r^2} \right)_{(z_0,0)} \quad (13.13)$$

322 Approximation of this term on numerical mesh has been already determined
 323 in previous subsection (13.6).

324 **13.8 Value of second partial derivative of V with respect to z on**
325 **axis Oz**

326 The third term of Laplace operator in rotationally symmetrical systems 13.2
327 takes form (on picture 10):

$$\left(\frac{\partial^2 V}{\partial z^2} \right)_{P_2} \approx \frac{\frac{V_3 - V_2}{h_z} - \frac{V_2 - V_1}{h_z}}{h_z} = \frac{V_1 + V_3 - 2V_2}{h_z^2} \quad (13.14)$$

328 Now we have determined all the 3 approximations o partial derivatives of V
329 in cylindrically symmetrical systems (on axis O_z).

330 **14 Partial derivatives off O_z axis**

331 **14.1 Personal note**

332 This is my personal interpretation. I cannot guarantee correctness of this ap-
333 proach

334 **14.2 Nodes numbering in Liebmann mesh (off axis O_z)**

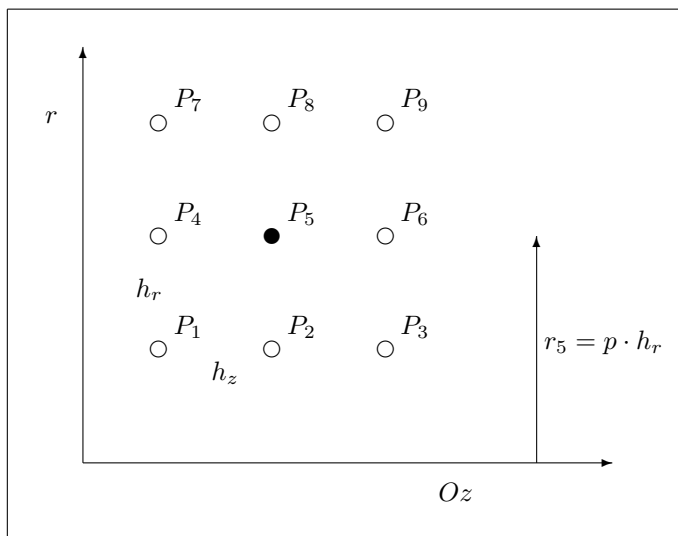


Figure 11: Nodes off axis O_z . Exemplary vector r_5 describes distance from axis O_z to node P_5

335 Mesh step in z direction is h_z . Mesh step in r direction is h_r . Sample mesh
336 points P_5 lies off O_z axis. Distance between mesh point P_5 and O_z axis is r_5 .

337 For ANSI C meshes (in Liebmann source code) the following relations have
338 place:

$$r = p h_r \quad (14.1)$$

$$p = i_{row} - 1 \quad (14.2)$$

339 **14.3 Taylor expansion of $V_{(z,r)}$ function in cylindrical coordinates**

340 Taylor expansion of $V_{(z,r)}$ function in cylindrical coordinates [3]:

$$\begin{aligned}
V_{(z,r)} = V_{(z_0,r_0)} + \left(\frac{\partial V}{\partial z} \right)_{(z_0,r_0)} (z - z_0) + \\
\left(\frac{\partial V}{\partial r} \right)_{(z_0,r_0)} (r - r_0) + \\
\frac{1}{2!} \left(\frac{\partial^2 V}{\partial z^2} \right)_{(z_0,r_0)} (z - z_0)^2 + \dots
\end{aligned} \tag{14.3}$$

341 **14.4 Laplace operator in rotationally symmetrical systems**

342 Laplace operator in cylindrical coordinates (cylindrical symmetry) consist of 3
343 elements [1] (on page 42):

$$\begin{aligned}
\nabla^2 (V_{(z,r)}) = \left(\frac{\partial^2 V}{\partial r^2} \right) + \\
\frac{1}{r} \left(\frac{\partial V}{\partial r} \right) + \\
\left(\frac{\partial^2 V}{\partial z^2} \right)
\end{aligned} \tag{14.4}$$

344 In this chapter we will try to determine approximation of each term.

345 **14.5 Value of second partial derivative of V with respect to r off**
346 **axis Oz**

$$\left(\frac{\partial^2 V}{\partial r^2} \right)_{P_5} \approx \frac{\frac{V_8 - V_5}{h_r} - \frac{V_5 - V_2}{h_r}}{h_r} = \frac{V_2 + V_8 - 2V_5}{h_r^2} \tag{14.5}$$

347 **14.6 Value of first partial derivative of V with respect to r divided**
348 **by r off axis Oz**

$$\frac{1}{r_5} \left(\frac{\partial V}{\partial r} \right)_{P_5} \approx \frac{1}{r_5} \frac{V_8 - V_2}{2h_r} = \frac{V_8 - V_2}{2r_5 h_z} \tag{14.6}$$

349 **14.7 Value of second partial derivative of V with respect to z off**
350 **axis Oz**

$$\left(\frac{\partial^2 V}{\partial z^2} \right)_{P_5} \approx \frac{\frac{V_6 - V_5}{h_z} - \frac{V_5 - V_4}{h_z}}{h_z} = \frac{V_4 + V_6 - 2V_5}{h_z^2} \tag{14.7}$$

351 **15 Relaxation formula for node P1 (on axis Oz)**

352 **15.1 Node description**

353 Left, bottom corner of mesh ZR (on axis Oz).

354 **15.2 Calculation of relaxation formula**

355 Laplace equation at node P_1

$$\nabla^2 (V_{(z,r)})_{P_1} = 0 \quad (15.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_1} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_1} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_1} = 0 \quad (15.2)$$

356 Approximation of partial derivatives of $V_{(z,r)}$ at node P_1

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_1} \approx \frac{\frac{V_4 - V_1}{h_r} - \frac{V_1 - V_4}{h_r}}{h_r} = \frac{2(V_4 - V_1)}{h_r^2} \quad (15.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_1} \approx \frac{2(V_4 - V_1)}{h_r^2} \quad (15.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_1} \approx \frac{\frac{V_2 - V_1}{h_z} - \frac{V_1 - V_{1z-}}{h_z}}{h_z} = \frac{V_2 - V_1}{h_z^2} - \frac{g_{1z-}}{h_z} \quad (15.5)$$

357 Let us substitute approximations to Laplace equation.

$$\frac{2(V_4 - V_1)}{h_r^2} + \frac{2(V_4 - V_1)}{h_r^2} + \frac{V_2 - V_1}{h_z^2} - \frac{g_{1z-}}{h_z} = 0 \quad (15.6)$$

$$\frac{2(V_4 - V_1)}{h_r^2} + \frac{2(V_4 - V_1)}{h_r^2} + \frac{V_2 - V_1}{h_z^2} = \frac{g_{1z-}}{h_z} \quad (15.7)$$

358 Let us find V_1

$$V_1 = ? \quad (15.8)$$

359 Let us multiply both sides

$$| \cdot h_z^2 h_r^2 \quad (15.9)$$

360 We obtain

$$2V_4 h_z^2 - 2V_1 h_z^2 + 2V_4 h_z^2 - 2V_1 h_z^2 + V_2 h_r^2 - V_1 h_r^2 = g_{1z-} h_z h_r^2 \quad (15.10)$$

361 Let us simplify this equation:

$$V_1 (2h_z^2 + 2h_z^2 + h_r^2) = 2V_4h_z^2 + 2V_4h_z^2 + V_2h_r^2 - g_{1z-}h_zh_r^2 \quad (15.11)$$

362 So we have:

$$V_1 (4h_z^2 + h_r^2) = 4V_4h_z^2 + V_2h_r^2 - g_{1z-}h_zh_r^2 \quad (15.12)$$

363 15.3 Final forms of relaxation formula

364 15.3.1 zrLV_RELAX5_P1_ON_A

$$\begin{aligned} h_z &\neq h_r \\ g_{1z-} &\neq 0 \\ V_1 &= \frac{4V_4h_z^2 + V_2h_r^2 - g_{1z-}h_zh_r^2}{4h_z^2 + h_r^2} \end{aligned} \quad (15.13)$$

365 15.3.2 zrLV_RELAX5_P1_ON_B

$$\begin{aligned} h_z &\neq h_r \\ g_{1z-} &= 0 \\ V_1 &= \frac{4V_4h_z^2 + V_2h_r^2}{4h_z^2 + h_r^2} \end{aligned} \quad (15.14)$$

366 15.3.3 zrLV_RELAX5_P1_ON_C

$$\begin{aligned} h_z &= h_r = h \\ g_{1z-} &\neq 0 \\ V_1 &= \frac{4V_4 + V_2 - g_{1z-}h}{5} \end{aligned} \quad (15.15)$$

367 15.3.4 zrLV_RELAX5_P1_ON_D

$$\begin{aligned} h_z &= h_r = h \\ g_{1z-} &= 0 \\ V_1 &= \frac{4V_4 + V_2}{5} \end{aligned} \quad (15.16)$$

368 **16 Relaxation formula for node P2 (on axis Oz)**

369 **16.1 Node description**

370 Bottom edge of mesh ZR (on axis Oz).

371 **16.2 Calculation of relaxation formula**

372 Laplace equation at node P_2

$$\nabla^2 (V_{(z,r)})_{P_2} = 0 \quad (16.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_2} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_2} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_2} = 0 \quad (16.2)$$

373 Approximation of partial derivatives of $V_{(z,r)}$ at node P_2

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_2} \approx \frac{\frac{V_5 - V_2}{h_r} - \frac{V_2 - V_5}{h_r}}{h_r} = \frac{2(V_5 - V_2)}{h_r^2} \quad (16.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_2} \approx \frac{2(V_5 - V_2)}{h_r^2} \quad (16.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_2} \approx \frac{\frac{V_3 - V_2}{h_z} - \frac{V_2 - V_1}{h_z}}{h_z} = \frac{V_1 + V_3 - 2V_2}{h_z^2} \quad (16.5)$$

374 Let us substitute approximations to Laplace equation.

$$\frac{2(V_5 - V_2)}{h_r^2} + \frac{2(V_5 - V_2)}{h_r^2} + \frac{V_1 + V_3 - 2V_2}{h_z^2} = 0 \quad (16.6)$$

375 There are no g expressions to move, to formula 7 has identical form as
376 formula 6.

$$\frac{2(V_5 - V_2)}{h_r^2} + \frac{2(V_5 - V_2)}{h_r^2} + \frac{V_1 + V_3 - 2V_2}{h_z^2} = 0 \quad (16.7)$$

377 Let us find V_2

$$V_2 = ? \quad (16.8)$$

378 Let us multiply both sides

$$| \cdot h_z^2 h_r^2 \quad (16.9)$$

379 We obtain

$$2V_5 h_z^2 - 2V_2 h_z^2 + 2V_5 h_r^2 - 2V_2 h_r^2 + V_1 h_r^2 + V_3 h_r^2 - 2V_2 h_r^2 = 0 \quad (16.10)$$

380 Let us simplify this equation:

$$V_2 (2h_z^2 + 2h_z^2 + 2h_r^2) = 2V_5 h_z^2 + 2V_5 h_z^2 + V_1 h_r^2 + V_3 h_r^2 \quad (16.11)$$

381 So we have:

$$V_2 (4h_z^2 + 2h_r^2) = 4V_5 h_z^2 + (V_1 + V_3) h_r^2 \quad (16.12)$$

382 16.3 Final forms of relaxation formula

383 16.3.1 zrLV_RELAX5_P2_ON_A

$$h_z \neq h_r$$

$$V_2 = \frac{4V_5 h_z^2 + (V_1 + V_3) h_r^2}{4h_z^2 + 2h_r^2} \quad (16.13)$$

384 16.3.2 zrLV_RELAX5_P2_ON_B

$$h_z \neq h_r$$

$$V_2 = \frac{4V_5 h_z^2 + (V_1 + V_3) h_r^2}{4h_z^2 + 2h_r^2} \quad (16.14)$$

385 16.3.3 zrLV_RELAX5_P2_ON_C

$$h_z = h_r = h$$

$$V_2 = \frac{4V_5 + V_1 + V_3}{6} \quad (16.15)$$

386 16.3.4 zrLV_RELAX5_P2_ON_D

$$h_z = h_r = h$$

$$V_2 = \frac{4V_5 + V_1 + V_3}{6} \quad (16.16)$$

387 **17 Relaxation formula for node P3 (on axis Oz)**

388 **17.1 Node description**

389 Right, bottom corner of mesh ZR (on axis Oz).

390 **17.2 Calculation of relaxation formula**

391 Laplace equation at node P_3

$$\nabla^2 (V_{(z,r)})_{P_3} = 0 \quad (17.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_3} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_3} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_3} = 0 \quad (17.2)$$

392 Approximation of partial derivatives of $V_{(z,r)}$ at node P_3

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_3} \approx \frac{\frac{V_6 - V_3}{h_r} - \frac{V_3 - V_6}{h_r}}{h_r} = \frac{2(V_6 - V_3)}{h_r^2} \quad (17.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_3} \approx \frac{2(V_6 - V_3)}{h_r^2} \quad (17.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_3} \approx \frac{\frac{V_{3z+} - V_3}{h_z} - \frac{V_3 - V_2}{h_z}}{h_z} = \frac{V_2 - V_3}{h_z^2} + \frac{g_{3z+}}{h_z} \quad (17.5)$$

393 Let us substitute approximations to Laplace equation.

$$\frac{2(V_6 - V_3)}{h_r^2} + \frac{2(V_6 - V_3)}{h_r^2} + \frac{V_2 - V_3}{h_z^2} + \frac{g_{3z+}}{h_z} = 0 \quad (17.6)$$

$$\frac{2(V_6 - V_3)}{h_r^2} + \frac{2(V_6 - V_3)}{h_r^2} + \frac{V_2 - V_3}{h_z^2} = -\frac{g_{3z+}}{h_z} \quad (17.7)$$

394 Let us find V_3

$$V_3 = ? \quad (17.8)$$

395 Let us multiply both sides

$$| \cdot h_z^2 h_r^2 \quad (17.9)$$

396 We obtain

$$2V_6 h_z^2 - 2V_3 h_z^2 + 2V_6 h_z^2 - 2V_3 h_z^2 + V_2 h_r^2 - V_3 h_r^2 = -g_{3z+} h_z h_r^2 \quad (17.10)$$

397 Let us simplify this equation:

$$V_3 (2h_z^2 + 2h_z^2 + h_r^2) = 2V_6h_z^2 + 2V_6h_z^2 + V_2h_r^2 + g_{3z+}h_zh_r^2 \quad (17.11)$$

398 So we have:

$$V_3 (4h_z^2 + h_r^2) = 4V_6h_z^2 + V_2h_r^2 + g_{1z-}h_zh_r^2 \quad (17.12)$$

399 17.3 Final forms of relaxation formula

400 17.3.1 zrLV_RELAX5_P3_ON_A

$$\begin{aligned} &h_z \neq h_r \\ &g_{3z+} \neq 0 \\ 401 \quad V_3 &= \frac{4V_6h_z^2 + V_2h_r^2 + g_{3z+}h_zh_r^2}{4h_z^2 + h_r^2} \end{aligned} \quad (17.13)$$

402 17.3.2 zrLV_RELAX5_P3_ON_B

$$\begin{aligned} &h_z \neq h_r \\ &g_{3z+} = 0 \\ V_3 &= \frac{4V_6h_z^2 + V_2h_r^2}{4h_z^2 + h_r^2} \end{aligned} \quad (17.14)$$

403 17.3.3 zrLV_RELAX5_P3_ON_C

$$\begin{aligned} &h_z = h_r = h \\ &g_{3z+} \neq 0 \\ V_3 &= \frac{4V_6 + V_2 + g_{3z+}h}{5} \end{aligned} \quad (17.15)$$

404 17.3.4 zrLV_RELAX5_P3_ON_D

$$\begin{aligned} &h_z = h_r = h \\ &g_{3z+} = 0 \\ V_3 &= \frac{4V_6 + V_2}{5} \end{aligned} \quad (17.16)$$

405 **18 Relaxation formula for node P4**

406 **18.1 Node description**

407 Left edge of mesh ZR.

408 **18.2 Calculation of relaxation formula**

409 Laplace equation at node P_4

$$\nabla^2 (V_{(z,r)})_{P_4} = 0 \quad (18.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_4} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_4} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_4} = 0 \quad (18.2)$$

410 Approximation of partial derivatives of $V_{(z,r)}$ at node P_4

411

412 (note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_4} \approx \frac{\frac{V_7 - V_4}{h_r} - \frac{V_4 - V_1}{h_r}}{h_r} = \frac{V_1 + V_7 - 2V_4}{h_r^2} \quad (18.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_4} \approx \frac{1}{r} \frac{V_7 - V_1}{2h_r} = \frac{V_7 - V_1}{2ph_r^2} \quad (18.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_4} \approx \frac{\frac{V_5 - V_1}{h_z} - \frac{V_4 - V_{4z-}}{h_z}}{h_z} = \frac{V_5 - V_4}{h_z^2} - \frac{g_{4z-}}{h_z} \quad (18.5)$$

413 Let us substitute approximations to Laplace equation.

$$\frac{V_1 + V_7 - 2V_4}{h_r^2} + \frac{V_7 - V_1}{2ph_r^2} + \frac{V_5 - V_4}{h_z^2} - \frac{g_{4z-}}{h_z} = 0 \quad (18.6)$$

$$\frac{V_1 + V_7 - 2V_4}{h_r^2} + \frac{V_7 - V_1}{2ph_r^2} + \frac{V_5 - V_4}{h_z^2} = \frac{g_{4z-}}{h_z} \quad (18.7)$$

414 Let us find V_4

$$V_4 = ? \quad (18.8)$$

415 Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \quad (18.9)$$

416 We obtain

$$2pV_1h_z^2 + 2pV_7h_z^2 - 4pV_4h_z^2 + V_7h_z^2 - V_1h_z^2 + 2pV_5h_r^2 - 2pV_4h_r^2 = 2pg_{4z-}h_zh_r^2 \quad (18.10)$$

417 Let us simplify this equation:

$$V_4(4ph_z^2 + 2ph_r^2) = V_1(2ph_z^2 - h_z^2) + V_7(2ph_z^2 + h_z^2) + V_52ph_r^2 - 2pg_{4z-}h_zh_r^2 \quad (18.11)$$

418 So we have:

$$V_42p(2h_z^2 + h_r^2) = V_1h_z^2(2p-1) + V_7h_z^2(2p+1) + V_52ph_r^2 - 2pg_{4z-}h_zh_r^2 \quad (18.12)$$

419 18.3 Final forms of relaxation formula

420 18.3.1 zrLV_RELAX5_P4_A

$$h_z \neq h_r$$

$$g_{4z-} \neq 0$$

$$421 V_4 = \frac{h_z^2(2p-1)}{2p(2h_z^2 + h_r^2)}V_1 + \frac{h_z^2(2p+1)}{2p(2h_z^2 + h_r^2)}V_7 + \frac{h_r^2}{2h_z^2 + h_r^2}V_5 - \frac{g_{4z-}h_zh_r^2}{2h_z^2 + h_r^2} \quad (18.13)$$

422 18.3.2 zrLV_RELAX5_P4_B

$$h_z \neq h_r$$

$$g_{4z-} = 0$$

$$423 V_4 = \frac{h_z^2(2p-1)}{2p(2h_z^2 + h_r^2)}V_1 + \frac{h_z^2(2p+1)}{2p(2h_z^2 + h_r^2)}V_7 + \frac{h_r^2}{2h_z^2 + h_r^2}V_5 \quad (18.14)$$

424 18.3.3 zrLV_RELAX5_P4_C

$$h_z = h_r = h$$

$$g_{4z-} \neq 0$$

$$425 V_4 = \frac{2p-1}{6p}V_1 + \frac{2p+1}{6p}V_7 + \frac{1}{3}V_5 - \frac{g_{4z-}h}{3} \quad (18.15)$$

426 18.3.4 zrLV_RELAX5_P4_D

$$h_z = h_r = h$$

$$g_{4z-} = 0$$

$$V_4 = \frac{2p-1}{6p}V_1 + \frac{2p+1}{6p}V_7 + \frac{1}{3}V_5 \quad (18.16)$$

427 **19 Relaxation formula for node P5**

428 **19.1 Node description**

429 Inner node of mesh ZR.

430 **19.2 Calculation of relaxation formula**

431 Laplace equation at node P_5

$$\nabla^2 (V_{(z,r)})_{P_5} = 0 \quad (19.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_5} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_5} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_5} = 0 \quad (19.2)$$

432 Approximation of partial derivatives of $V_{(z,r)}$ at node P_5

433

434 (note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_5} \approx \frac{\frac{V_8 - V_5}{h_r} - \frac{V_5 - V_2}{h_r}}{h_r} = \frac{V_2 + V_8 - 2V_5}{h_r^2} \quad (19.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_5} \approx \frac{1}{r} \frac{V_8 - V_2}{2h_r} = \frac{V_8 - V_2}{2ph_r^2} \quad (19.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_5} \approx \frac{\frac{V_6 - V_5}{h_z} - \frac{V_5 - V_4}{h_z}}{h_z} = \frac{V_4 + V_6 - 2V_5}{h_z^2} \quad (19.5)$$

435 Let us substitute approximations to Laplace equation.

$$\frac{V_2 + V_8 - 2V_5}{h_r^2} + \frac{V_8 - V_2}{2ph_r^2} + \frac{V_4 + V_6 - 2V_5}{h_z^2} = 0 \quad (19.6)$$

436 We don't need to simplify this equation in step 7:

$$\frac{V_2 + V_8 - 2V_5}{h_r^2} + \frac{V_8 - V_2}{2ph_r^2} + \frac{V_4 + V_6 - 2V_5}{h_z^2} = 0 \quad (19.7)$$

437 Let us find V_5

$$V_5 = ? \quad (19.8)$$

438 Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \quad (19.9)$$

439 We obtain

$$2pV_2h_z^2 + 2pV_8h_z^2 - 4pV_5h_z^2 + V_8h_z^2 - V_2h_z^2 + 2pV_4h_r^2 + 2pV_6h_r^2 - 4pV_5h_r^2 = 0 \quad (19.10)$$

440 Let us simplify this equation:

$$V_5(4ph_z^2 + 4ph_r^2) = V_2(2ph_z^2 - h_z^2) + V_8(2ph_z^2 + h_z^2) + 2ph_r^2V_4 + 2ph_r^2V_6 \quad (19.11)$$

441 So we have:

$$V_54p(h_z^2 + h_r^2) = V_2h_z^2(2p - 1) + V_8h_z^2(2p + 1) + V_42ph_r^2 + V_62ph_r^2 \quad (19.12)$$

442 19.3 Final forms of relaxation formula

443 19.3.1 zrLV_RELAX5_P5_A

$$444 \quad h_z \neq h_r$$

$$V_5 = \frac{h_z^2(2p - 1)}{4p(h_z^2 + h_r^2)}V_2 + \frac{h_z^2(2p + 1)}{4p(h_z^2 + h_r^2)}V_8 + \frac{h_r^2}{2(h_z^2 + h_r^2)}(V_4 + V_6) \quad (19.13)$$

445 19.3.2 zrLV_RELAX5_P5_B

446 This formula is identical to formula A:

$$447 \quad h_z \neq h_r$$

$$V_5 = \frac{h_z^2(2p - 1)}{4p(h_z^2 + h_r^2)}V_2 + \frac{h_z^2(2p + 1)}{4p(h_z^2 + h_r^2)}V_8 + \frac{h_r^2}{2(h_z^2 + h_r^2)}(V_4 + V_6) \quad (19.14)$$

448 19.3.3 zrLV_RELAX5_P5_C

$$449 \quad h_z = h_r = h$$

$$V_5 = \frac{2p - 1}{8p}V_2 + \frac{2p + 1}{8p}V_8 + \frac{1}{4}(V_4 + V_6) \quad (19.15)$$

450 19.3.4 zrLV_RELAX5_P5_D

451 This formula is identical to formula C:

$$452 \quad h_z = h_r = h$$

$$V_5 = \frac{2p - 1}{8p}V_2 + \frac{2p + 1}{8p}V_8 + \frac{1}{4}(V_4 + V_6) \quad (19.16)$$

453 **20 Relaxation formula for node P6**

454 **20.1 Node description**

455 Right edge of mesh ZR.

456 **20.2 Calculation of relaxation formula**

457 Laplace equation at node P_6

$$\nabla^2 (V_{(z,r)})_{P_6} = 0 \quad (20.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_6} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_6} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_6} = 0 \quad (20.2)$$

458 Approximation of partial derivatives of $V_{(z,r)}$ at node P_6

459

460 (note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_6} \approx \frac{\frac{V_9 - V_6}{h_r} - \frac{V_6 - V_3}{h_r}}{h_r} = \frac{V_3 + V_9 - 2V_6}{h_r^2} \quad (20.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_6} \approx \frac{1}{r} \frac{V_9 - V_3}{2h_r} = \frac{V_9 - V_3}{2ph_r^2} \quad (20.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_6} \approx \frac{\frac{V_{6z+} - V_6}{h_z} - \frac{V_6 - V_5}{h_z}}{h_z} = \frac{V_5 - V_6}{h_z^2} + \frac{g_{6z+}}{h_z} \quad (20.5)$$

461 Let us substitute approximations to Laplace equation.

$$\frac{V_3 + V_9 - 2V_6}{h_r^2} + \frac{V_9 - V_3}{2ph_r^2} + \frac{V_5 - V_6}{h_z^2} + \frac{g_{6z+}}{h_z} = 0 \quad (20.6)$$

$$\frac{V_3 + V_9 - 2V_6}{h_r^2} + \frac{V_9 - V_3}{2ph_r^2} + \frac{V_5 - V_6}{h_z^2} = -\frac{g_{6z+}}{h_z} \quad (20.7)$$

462 Let us find V_6

$$V_6 = ? \quad (20.8)$$

463 Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \quad (20.9)$$

464 We obtain

$$2pV_3h_z^2 + 2pV_9h_z^2 - 4pV_6h_z^2 + V_9h_z^2 - V_3h_z^2 + 2pV_5h_r^2 - 2pV_6h_r^2 = -2pg_{6z+}h_zh_r^2 \quad (20.10)$$

465 Let us simplify this equation:

$$V_6(4ph_z^2 + 2ph_r^2) = V_3(2ph_z^2 - h_z^2) + V_9(2ph_z^2 + h_z^2) + V_52ph_r^2 + 2pg_{6z+}h_zh_r^2 \quad (20.11)$$

466 So we have:

$$V_62p(2h_z^2 + h_r^2) = V_3h_z^2(2p-1) + V_9h_z^2(2p+1) + V_52ph_r^2 + 2pg_{6z+}h_zh_r^2 \quad (20.12)$$

467 20.3 Final forms of relaxation formula

468 20.3.1 zrLV_RELAX5_P6_A

$$h_z \neq h_r \\ g_{6z+} \neq 0 \\ 469 V_6 = \frac{h_z^2(2p-1)}{2p(2h_z^2 + h_r^2)}V_3 + \frac{h_z^2(2p+1)}{2p(2h_z^2 + h_r^2)}V_9 + \frac{h_r^2}{2h_z^2 + h_r^2}V_5 + \frac{g_{6z+}h_zh_r^2}{2h_z^2 + h_r^2} \quad (20.13)$$

470 20.3.2 zrLV_RELAX5_P6_B

$$h_z \neq h_r \\ g_{6z+-} = 0 \\ 471 V_6 = \frac{h_z^2(2p-1)}{2p(2h_z^2 + h_r^2)}V_3 + \frac{h_z^2(2p+1)}{2p(2h_z^2 + h_r^2)}V_9 + \frac{h_r^2}{2h_z^2 + h_r^2}V_5 \quad (20.14)$$

472 20.3.3 zrLV_RELAX5_P6_C

$$h_z = h_r = h \\ g_{6z+} \neq 0 \\ 473 V_6 = \frac{2p-1}{6p}V_3 + \frac{2p+1}{6p}V_9 + \frac{1}{3}V_5 + \frac{g_{6z+}h}{3} \quad (20.15)$$

474 20.3.4 zrLV_RELAX5_P6_D

$$h_z = h_r = h \\ g_{6z+} = 0 \\ 475 V_4 = \frac{2p-1}{6p}V_3 + \frac{2p+1}{6p}V_9 + \frac{1}{3}V_5 \quad (20.16)$$

476 **21 Relaxation formula for node P7**

477 **21.1 Node description**

478 Left, upper corner of mesh ZR.

479 **21.2 Calculation of relaxation formula**

480 Laplace equation at node P_7

$$\nabla^2 (V_{(z,r)})_{P_7} = 0 \quad (21.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_7} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_7} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_7} = 0 \quad (21.2)$$

481 Approximation of partial derivatives of $V_{(z,r)}$ at node P_7

482

483 (note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_7} \approx \frac{\frac{V_{7r+} - V_7}{h_r} - \frac{V_7 - V_4}{h_r}}{h_r} = \frac{V_4 - V_7}{h_r^2} + \frac{g_{7r+}}{h_r} \quad (21.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_7} \approx \frac{1}{r} \frac{V_{7r+} - V_4}{2h_r} = \frac{V_7 + g_{7r+}h_r - V_4}{2ph_r^2} = \frac{V_7 - V_4}{2ph_r^2} + \frac{g_{7r+}}{2ph_r} \quad (21.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_7} \approx \frac{\frac{V_8 - V_7}{h_z} - \frac{V_7 - V_{7z-}}{h_z}}{h_z} = \frac{V_8 - V_7}{h_z^2} - \frac{g_{7z-}}{h_z} \quad (21.5)$$

484 Let us substitute approximations to Laplace equation.

$$\frac{V_4 - V_7}{h_r^2} + \frac{g_{7r+}}{h_r} + \frac{V_7 - V_4}{2ph_r^2} + \frac{g_{7r+}}{2ph_r} + \frac{V_8 - V_7}{h_z^2} - \frac{g_{7z-}}{h_z} = 0 \quad (21.6)$$

$$\frac{V_4 - V_7}{h_r^2} + \frac{V_7 - V_4}{2ph_r^2} + \frac{V_8 - V_7}{h_z^2} = -\frac{g_{7r+}}{h_r} - \frac{g_{7r+}}{2ph_r} + \frac{g_{7z-}}{h_z} \quad (21.7)$$

485 Let us find V_7

$$V_7 = ? \quad (21.8)$$

486 Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \quad (21.9)$$

487 We obtain

$$2pV_4 h_z^2 - 2pV_7 h_z^2 + V_7 h_z^2 - V_4 h_z^2 + 2pV_8 h_r^2 - 2pV_7 h_r^2 = \quad (21.10)$$

$$-2pg_{7r+} h_z^2 h_r - g_{7r+} h_z^2 h_r + 2pg_{7z-} h_z h_r^2$$

488 Let us simplify this equation:

$$V_7 (2ph_z^2 - h_z^2 + 2ph_r^2) = V_4 (2ph_z^2 - h_z^2) + V_8 (2ph_r^2) + \quad (21.11)$$

$$2pg_{7r+} h_z^2 h_r + g_{7r+} h_z^2 h_r - 2pg_{7z-} h_z h_r^2$$

489 So we have:

$$V_7 ((2p-1)h_z^2 + 2ph_r^2) = V_4 h_z^2 (2p-1) + V_8 2ph_r^2 + \quad (21.12)$$

$$2pg_{7r+} h_z^2 h_r + g_{7r+} h_z^2 h_r - 2pg_{7z-} h_z h_r^2$$

490 21.3 Final forms of relaxation formula

491 21.3.1 zrLV_RELAX5_P7_A

$$h_z \neq h_r$$

$$g_{7z-} \neq 0$$

$$g_{7r+} \neq 0$$

492

$$V_7 = \frac{h_z^2 (2p-1)}{(2p-1)h_z^2 + 2ph_r^2} V_4 + \frac{2ph_r^2}{(2p-1)h_z^2 + 2ph_r^2} V_8 + \quad (21.13)$$

$$\frac{(2p+1)g_{7r+} h_z^2 h_r - 2pg_{7z-} h_z h_r^2}{(2p-1)h_z^2 + 2ph_r^2}$$

493 21.3.2 zrLV_RELAX5_P7_B

$$h_z \neq h_r$$

$$g_{7z-} = 0$$

$$g_{7r+} = 0$$

$$V_7 = \frac{h_z^2 (2p-1)}{(2p-1)h_z^2 + 2ph_r^2} V_4 + \frac{2ph_r^2}{(2p-1)h_z^2 + 2ph_r^2} V_8 \quad (21.14)$$

494 **21.3.3 zrLV_RELAX5_P7_C**

$$h_z = h_r = h$$

$$g_{7z-} \neq 0$$

$$g_{7r+} \neq 0$$

495

$$V_7 = \frac{2p-1}{4p-1}V_4 + \frac{2p}{4p-1}V_8 + \frac{h((2p+1)g_{7r+} - g_{7z-})}{4p-1} \quad (21.15)$$

496 **21.3.4 zrLV_RELAX5_P7_D**

$$h_z = h_r = h$$

$$g_{7z-} = 0$$

$$g_{7r+} = 0$$

497

$$V_7 = \frac{2p-1}{4p-1}V_4 + \frac{2p}{4p-1}V_8 \quad (21.16)$$

498 **22 Relaxation formula for node P8**

499 **22.1 Node description**

500 Upper edge of mesh ZR.

501 **22.2 Calculation of relaxation formula**

502 Laplace equation at node P_8

$$\nabla^2 (V_{(z,r)})_{P_8} = 0 \quad (22.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_8} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_8} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_8} = 0 \quad (22.2)$$

503 Approximation of partial derivatives of $V_{(z,r)}$ at node P_8

504

505 (note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_8} \approx \frac{\frac{V_{8r+} - V_8}{h_r} - \frac{V_8 - V_5}{h_r}}{h_r} = \frac{V_5 - V_8}{h_r^2} + \frac{g_{8r+}}{h_r} \quad (22.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_8} \approx \frac{1}{r} \frac{V_{8r+} - V_5}{2h_r} = \frac{V_8 + g_{8r+}h_r - V_5}{2ph_r^2} = \frac{V_8 - V_5}{2ph_r^2} + \frac{g_{8r+}}{2ph_r} \quad (22.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_8} \approx \frac{\frac{V_9 - V_8}{h_z} - \frac{V_8 - V_7}{h_z}}{h_z} = \frac{V_7 + V_9 - 2V_8}{h_z^2} \quad (22.5)$$

506 Let us substitute approximations to Laplace equation.

$$\frac{V_5 - V_8}{h_r^2} + \frac{g_{8r+}}{h_r} + \frac{V_8 - V_5}{2ph_r^2} + \frac{g_{8r+}}{2ph_r} + \frac{V_7 + V_9 - 2V_8}{h_z^2} = 0 \quad (22.6)$$

$$\frac{V_5 - V_8}{h_r^2} + \frac{V_8 - V_5}{2ph_r^2} + \frac{V_7 + V_9 - 2V_8}{h_z^2} = -\frac{g_{8r+}}{h_r} - \frac{g_{8r+}}{2ph_r} \quad (22.7)$$

507 Let us find V_8

$$V_8 = ? \quad (22.8)$$

508 Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \quad (22.9)$$

509 We obtain

$$2pV_5 h_z^2 - 2pV_8 h_z^2 + V_8 h_z^2 - V_5 h_z^2 + 2pV_7 h_r^2 + 2pV_9 h_r^2 - 4pV_8 h_r^2 = -2pg_{8r+} h_z^2 h_r - g_{8r+} h_z^2 h_r \quad (22.10)$$

510 Let us simplify this equation:

$$V_8 (2ph_z^2 - h_z^2 + 4ph_r^2) = V_5 (2ph_z^2 - h_z^2) + (V_7 + V_9) 2ph_r^2 + 2pg_{8r+} h_z^2 h_r + g_{8r+} h_z^2 h_r \quad (22.11)$$

511 So we have:

$$V_8 ((2p-1)h_z^2 + 4ph_r^2) = V_5 h_z^2 (2p-1) + (V_7 + V_9) 2ph_r^2 + 2pg_{8r+} h_z^2 h_r + g_{8r+} h_z^2 h_r \quad (22.12)$$

512 22.3 Final forms of relaxation formula

513 22.3.1 zrLV_RELAX5_P8_A

$$h_z \neq h_r$$

$$g_{8r+} \neq 0$$

$$V_8 = \frac{h_z^2 (2p-1)}{(2p-1)h_z^2 + 4ph_r^2} V_5 + \frac{2ph_r^2}{(2p-1)h_z^2 + 4ph_r^2} (V_7 + V_9) + \frac{(2p+1)h_z^2 h_r g_{8r+}}{(2p-1)h_z^2 + 4ph_r^2} \quad (22.13)$$

515 22.3.2 zrLV_RELAX5_P8_B

$$h_z \neq h_r$$

$$g_{8r+} = 0$$

$$V_8 = \frac{h_z^2 (2p-1)}{(2p-1)h_z^2 + 4ph_r^2} V_5 + \frac{2ph_r^2}{(2p-1)h_z^2 + 4ph_r^2} (V_7 + V_9) \quad (22.14)$$

517 22.3.3 zrLV_RELAX5_P8_C

$$h_z = h_r = h$$

$$g_{8r+} \neq 0$$

$$V_8 = \frac{2p-1}{6p-1} V_5 + \frac{2p}{6p-1} (V_7 + V_9) + \frac{(2p+1)h g_{8r+}}{6p-1} \quad (22.15)$$

519 **22.3.4 zrLV_RELAX5_P8_D**

$$h_z = h_r = h$$

$$g_{8r+} = 0$$

520

$$V_8 = \frac{2p-1}{6p-1}V_5 + \frac{2p}{6p-1}(V_7 + V_9) \quad (22.16)$$

521 **23 Relaxation formula for node P9**

522 **23.1 Node description**

523 Right, upper corner of mesh ZR.

524 **23.2 Calculation of relaxation formula**

525 Laplace equation at node P_9

$$\nabla^2 (V_{(z,r)})_{P_9} = 0 \quad (23.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_9} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_9} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_9} = 0 \quad (23.2)$$

526 Approximation of partial derivatives of $V_{(z,r)}$ at node P_9

527

528 (note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_9} \approx \frac{V_{9r+} - V_9}{h_r} - \frac{V_9 - V_6}{h_r} = \frac{V_6 - V_9}{h_r^2} + \frac{g_{9r+}}{h_r} \quad (23.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_9} \approx \frac{1}{r} \frac{V_{9r+} - V_6}{2h_r} = \frac{V_9 + g_{9r+}h_r - V_6}{2ph_r^2} = \frac{V_9 - V_6}{2ph_r^2} + \frac{g_{9r+}}{2ph_r} \quad (23.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_9} \approx \frac{V_{9z+} - V_9}{h_z} - \frac{V_9 - V_8}{h_z} = \frac{g_{9z+}}{h_z} + \frac{V_8 - V_9}{h_z^2} \quad (23.5)$$

529 Let us substitute approximations to Laplace equation.

$$\frac{V_6 - V_9}{h_r^2} + \frac{g_{9r+}}{h_r} + \frac{V_9 - V_6}{2ph_r^2} + \frac{g_{9r+}}{2ph_r} + \frac{V_8 - V_9}{h_z^2} + \frac{g_{9z+}}{h_z} = 0 \quad (23.6)$$

$$\frac{V_6 - V_9}{h_r^2} + \frac{V_9 - V_6}{2ph_r^2} + \frac{V_8 - V_9}{h_z^2} = -\frac{g_{9r+}}{h_r} - \frac{g_{9r+}}{2ph_r} - \frac{g_{9z+}}{h_z} \quad (23.7)$$

530 Let us find V_9

$$V_9 = ? \quad (23.8)$$

531 Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \quad (23.9)$$

532 We obtain

$$2pV_6 h_z^2 - 2pV_9 h_z^2 + V_9 h_z^2 - V_6 h_z^2 + 2pV_8 h_r^2 - 2pV_9 h_r^2 = \quad (23.10)$$

$$-2pg_{9r+} h_z^2 h_r - g_{9r+} h_z^2 h_r - 2pg_{9z+} h_z h_r^2$$

533 Let us simplify this equation:

$$V_9 (2ph_z^2 - h_z^2 + 2ph_r^2) = V_6 (2ph_z^2 - h_z^2) + V_8 (2ph_r^2) + \quad (23.11)$$

$$2pg_{9r+} h_z^2 h_r + g_{9r+} h_z^2 h_r + 2pg_{9z+} h_z h_r^2$$

534 So we have:

$$V_9 ((2p-1)h_z^2 + 2ph_r^2) = V_6 h_z^2 (2p-1) + V_8 2ph_r^2 + \quad (23.12)$$

$$(2p+1)g_{9r+} h_z^2 h_r + 2pg_{9z+} h_z h_r^2$$

535 23.3 Final forms of relaxation formula

536 23.3.1 zrLV_RELAX5_P9_A

$$h_z \neq h_r$$

$$g_{9z-} \neq 0$$

$$g_{9r+} \neq 0$$

537

$$V_9 = \frac{h_z^2 (2p-1)}{(2p-1)h_z^2 + 2ph_r^2} V_6 + \frac{2ph_r^2}{(2p-1)h_z^2 + 2ph_r^2} V_8 + \quad (23.13)$$

$$\frac{(2p+1)g_{9r+} h_z^2 h_r + 2pg_{9z+} h_z h_r^2}{(2p-1)h_z^2 + 2ph_r^2}$$

538 23.3.2 zrLV_RELAX5_P9_B

$$h_z \neq h_r$$

$$g_{9z-} = 0$$

$$g_{9r+} = 0$$

$$V_9 = \frac{h_z^2 (2p-1)}{(2p-1)h_z^2 + 2ph_r^2} V_6 + \frac{2ph_r^2}{(2p-1)h_z^2 + 2ph_r^2} V_8 \quad (23.14)$$

539 **23.3.3 zrLV_RELAX5_P9_C**

$$h_z = h_r = h$$

$$g_{9z-} \neq 0$$

$$g_{9r+} \neq 0$$

540

$$V_9 = \frac{2p-1}{4p-1}V_6 + \frac{2p}{4p-1}V_8 + \frac{h((2p+1)g_{9r+} + g_{9z+})}{4p-1} \quad (23.15)$$

541 **23.3.4 zrLV_RELAX5_P9_D**

$$h_z = h_r = h$$

$$g_{9z-} = 0$$

$$g_{9r+} = 0$$

542

$$V_9 = \frac{2p-1}{4p-1}V_6 + \frac{2p}{4p-1}V_8 \quad (23.16)$$

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