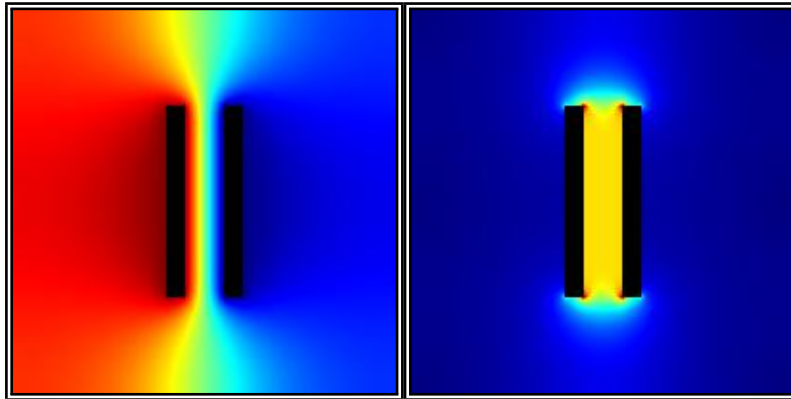


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Liebmann technical documentation



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Laplace equation 2D (XY)
(Cartesian coordinates)
relaxation scheme explained
(5 - point star)

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project homepage: http://marcinkulbaka.prv.pl/Liebmann/index_en.html

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version 10

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2024.09.03

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University of Maria Curie - Skłodowska in Lublin, Poland

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99 **1 Liebmann technical documentation series**

- 100 1. Wyznaczanie rozkładu pola elektrostatycznego w próżni metodą relak-
101 sacyjną Liebmann. (Polish version / wersja polska)
- 102 2. Determination of electrostatic field distribution by using Liebmann relax-
103 ation method. (English version / wersja angielska)
- 104 3. Graphics. Mapping voltages to colours (colormaps).
- 105 4. Laplace equation 2D (XY). (Cartesian coordinates). Relaxation scheme
106 explained. (5 - point star)
- 107 5. Laplace equation 2D (ZR) (Cylindrical coordinates). Relaxation scheme
108 explained. (5 - point star)
- 109 6. Liebmann source code. (ANSI C programming language)

110 **2 Versions of this document**

- 111 1. version 1 - 2023.11.03
- 112 2. version 2 - 2024.01.26
- 113 3. version 3 - 2024.02.02
- 114 4. version 4 - 2024.02.05
- 115 5. version 5 - 2024.05.18
- 116 6. version 6 - 2024.05.23
- 117 7. version 7 - 2024.05.24
- 118 8. version 8 - 2024.07.17
- 119 9. version 9 - 2024.07.18
- 120 10. version 10 - 2024.09.03

121 **3 Solving Laplace equation using relaxation method**

122 I tried to solve Laplace equation using mainly information from Pierre Grivet's
123 book (Electron Optics) - [1].
124 There are few editions of this book (1965, 1972). Second edition (1972) con-
125 tains explanation of relaxation method (page 38).

126 More generalized approaches has been drafted by James R. Nagel - [2].
127 <https://my.ece.utah.edu/~ece6340/LECTURES/Feb1/> (visited 2023-03-01).

128

129 There are also publications edited by Albert Septier: Focusing of Charged
130 Particles [3] and Applied Charged Particle Optics (part A). [4].

131 I have also found some ideas in publication of D W O Heddle: Electrostatic
132 Lens Systems [5] (especially using PC computers to solve electrostatic prob-
133 lems).

134 I have also found (brief) description of by - hand solving of Laplace equa-
135 tion by Bohdan Paszkowski - [6] (Polish edition). English translation of this book
136 also exists - [7].

137

138 I would like to thank many people, who helped me with this challenge. Espe-
139 cially prof. dr hab. Mieczysław Jałochowski (supervisor of my master's thesis),
140 who enabled me to use SIMION and MATLAB software while writing master's
141 thesis about electron optical systems at University of Maria Curie - Skłodowska
142 in Lublin in 2008. I would also thank to prof. Marcin Turek for fruitful discus-
143 sion about numerical methods. What is more, my colleague Bartosz in 2012
144 had explained me general problems with software efficiency. So he had also
145 contributed significantly to the idea of Liebmann software (especially using C
146 language).

147 4 Explanation of symbols in calculations

148 • P_i - i -th mesh node

149 • V_i - value of electrostatic potential at node P_i . Unit - [V]

150 • h - mesh step (for example h_x - mesh step in x direction). Unit - [mm]

151 • $g_{i+/-}$ - gradient in direction i (for example $g_{1x-} = \frac{V_1 - V_{1x-}}{h_x}$. Unit - [$\frac{V}{mm}$])

152 • i_{row} - index of row in mesh. Values of $i_{row} = 1, 2, \dots, \text{size_row}$

153 • i_{col} - index of column in mesh. Values of $i_{col} = 1, 2, \dots, \text{size_col}$

154 Symbols in final relaxation formulae

155 xyLV_RELAX5_P1_A

156 • xy - coordinates (2D, planar)

157 • LV - Laplace equation in vacuum (no dielectrics)

158 • RELAX_5 - 5- point relaxation method

159 • P1 - relaxation scheme for point P1 (in general P1 .. P9)

160 • A - mesh type A (in general A .. D)

161 **5 Mesh XY - type A**

162 $h_x \neq h_y$

163 gradient V outside a mesh exists

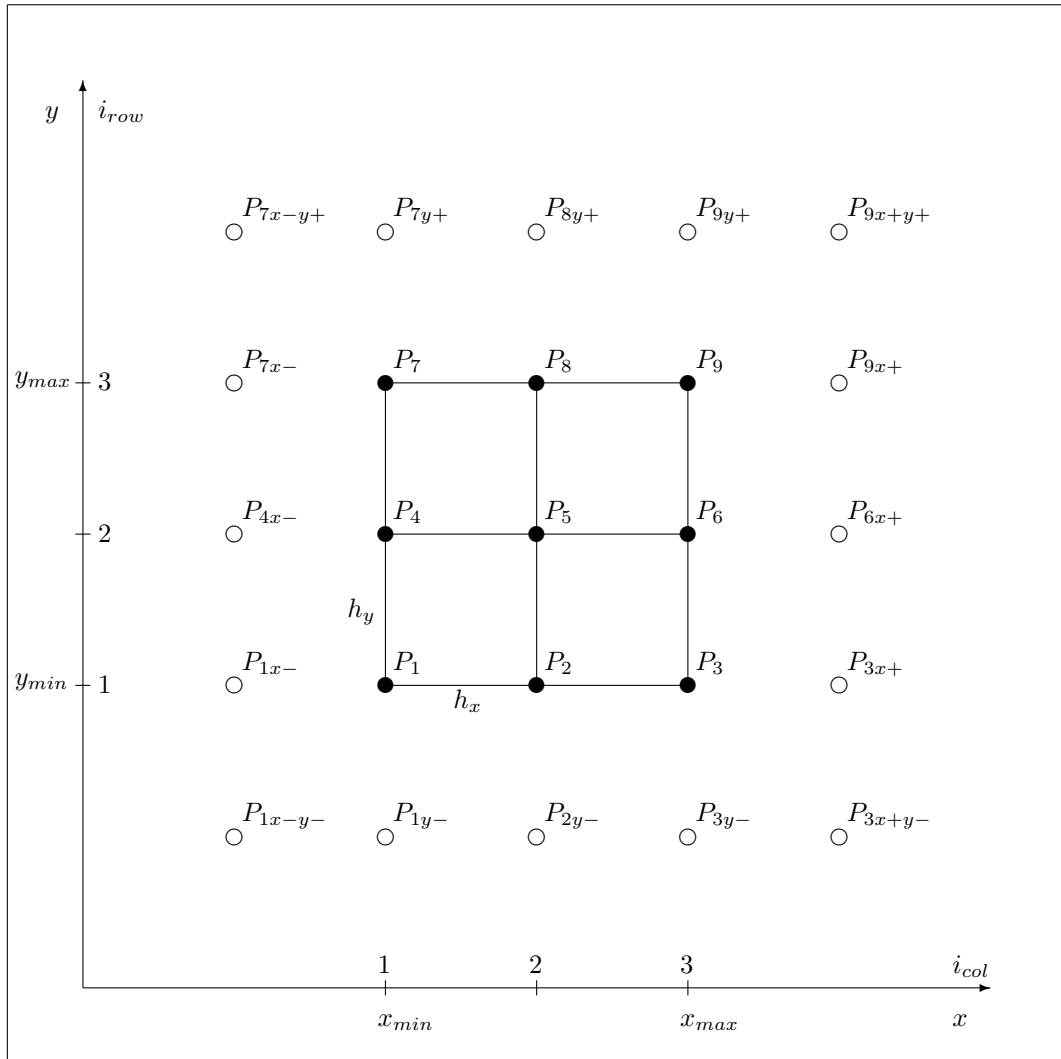


Figure 1: Mesh XY type A

164 **6 Mesh XY - type B**

165 $h_x \neq h_y$

166 gradient V outside a mesh does not exist

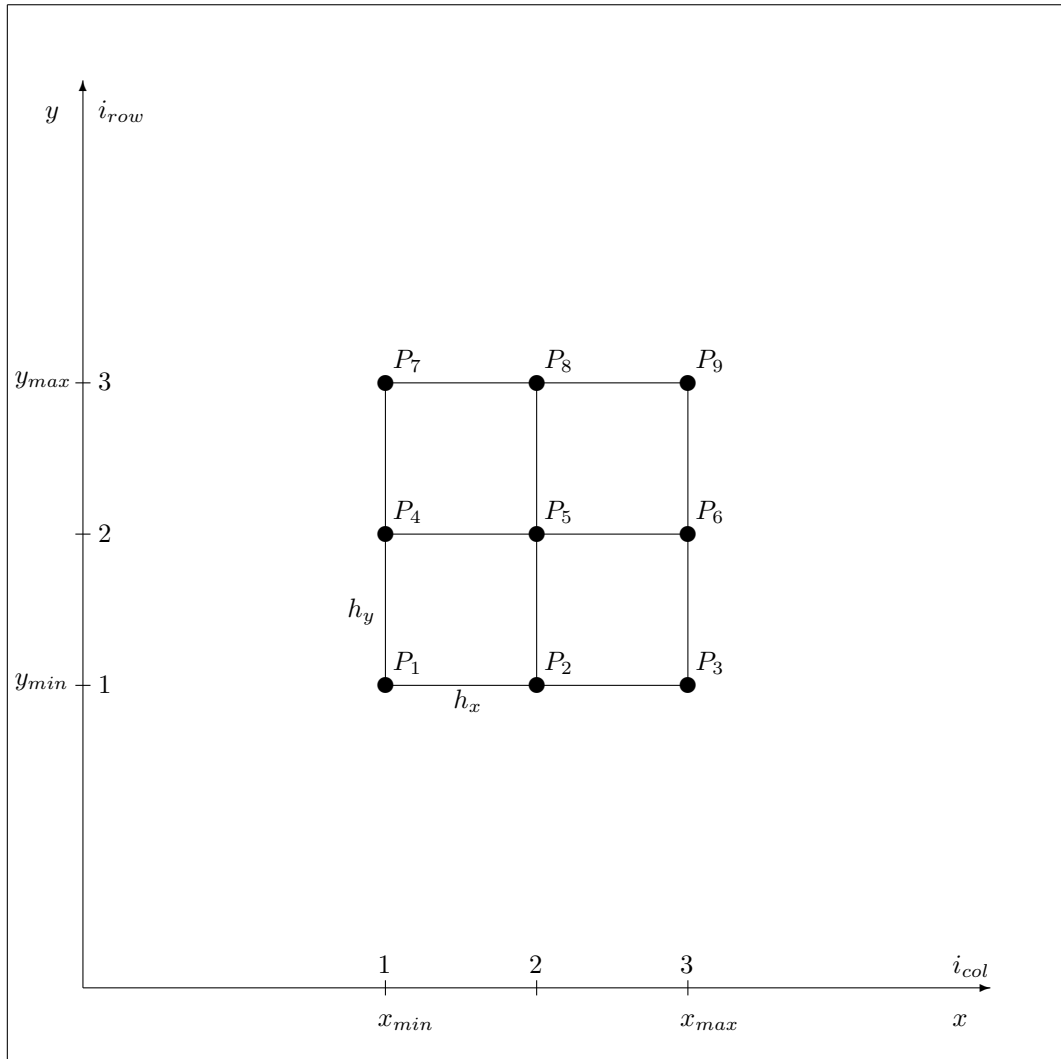


Figure 2: Mesh XY type B

167 **7 Mesh XY - type C**

168 $h_x = h_y = h$

169 gradient V outside a mesh exists

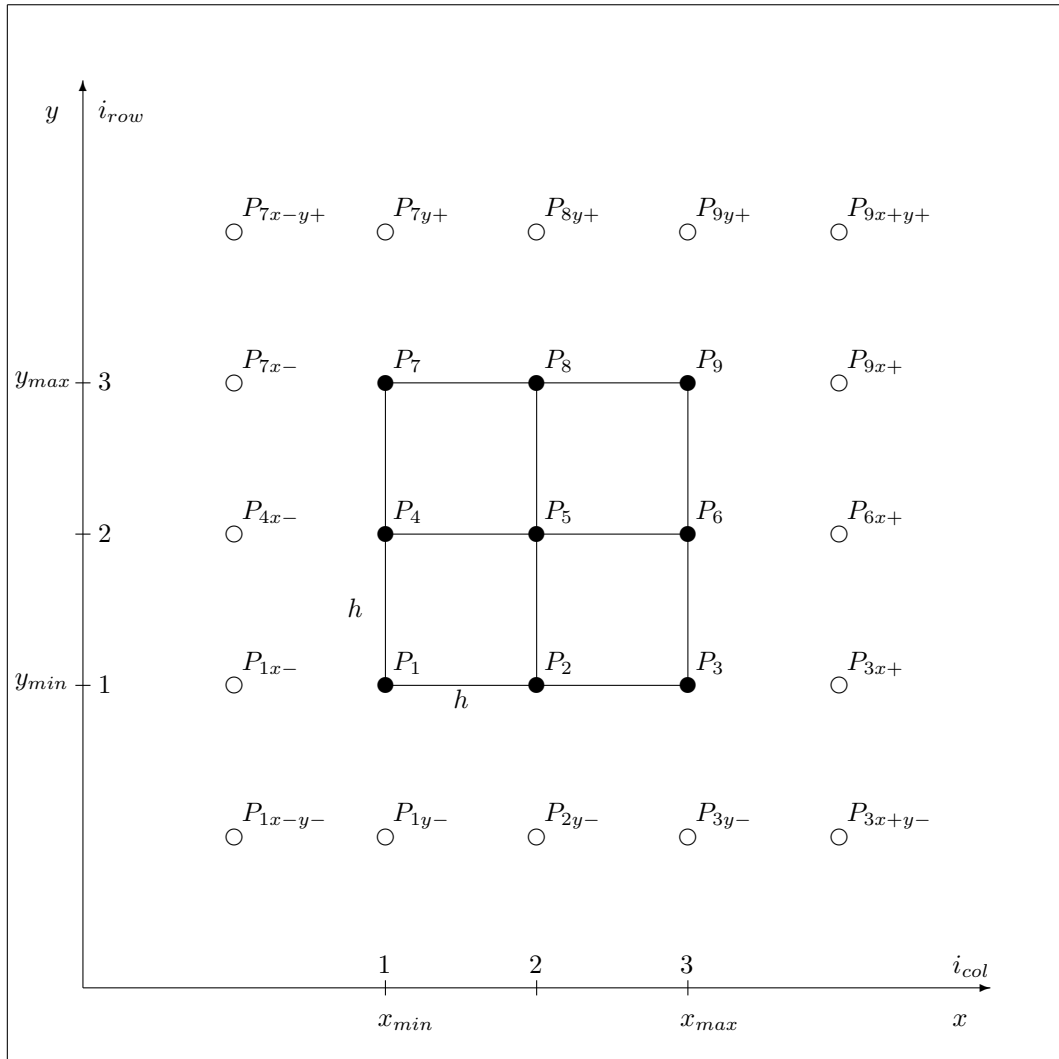


Figure 3: Mesh XY type C

170 **8 Mesh XY - type D**

171 $h_x = h_y = h$

172 gradient V outside a mesh does not exist

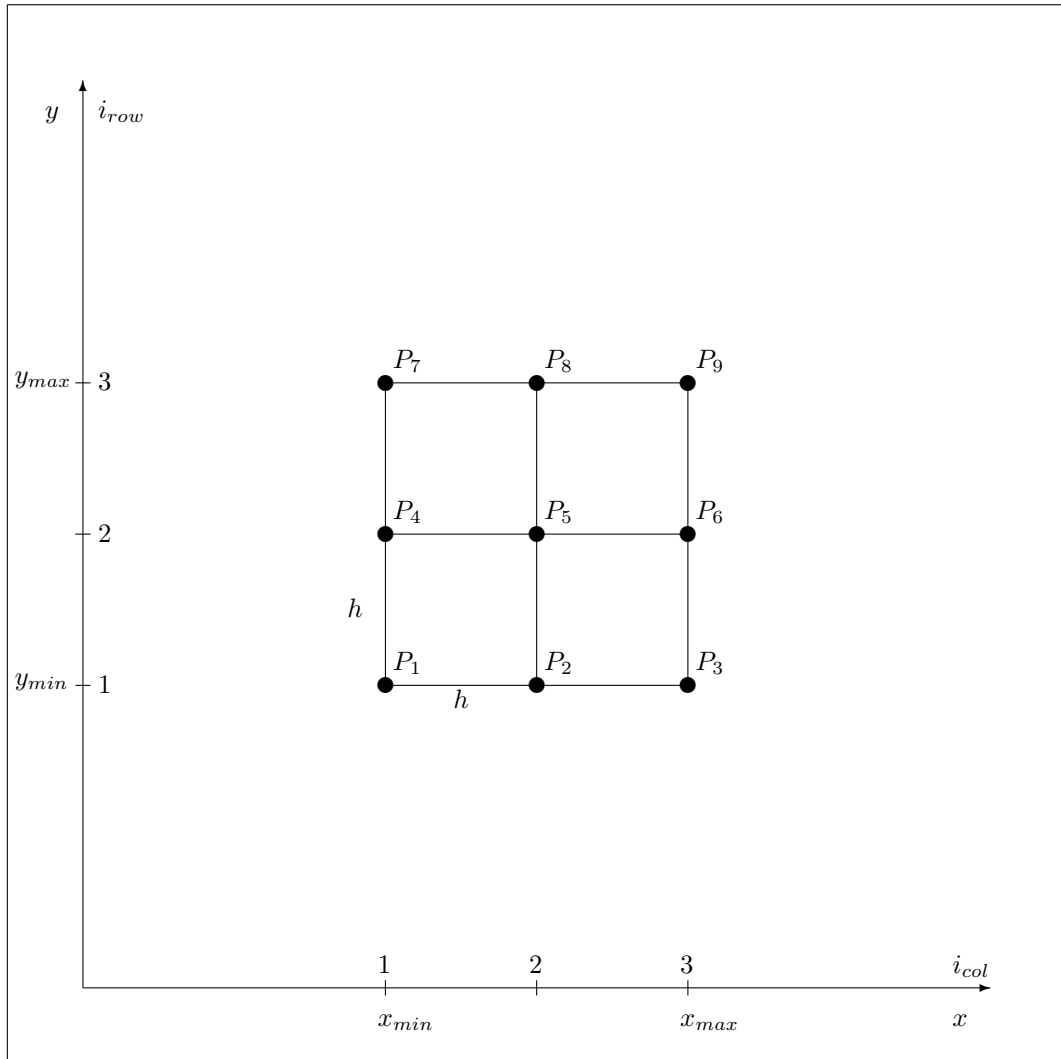


Figure 4: Mesh XY type D

173 **9 Example of A-type mesh in ANSI C**

174 Example of A- type mesh in ANSI C program. The mesh is represented by 2
 175 dimensional array of double precision numbers. Rows and columns in mesh
 176 are numbered from 1 (this was my choice) instead of default 0 (as usual in C
 177 language). This choice has pros and cons. It is easier to calculate mesh size
 178 (size_row * size_col). Access to each node can be also more intuitive, but logic
 179 in each library function must contain this shift between node ordering styles.

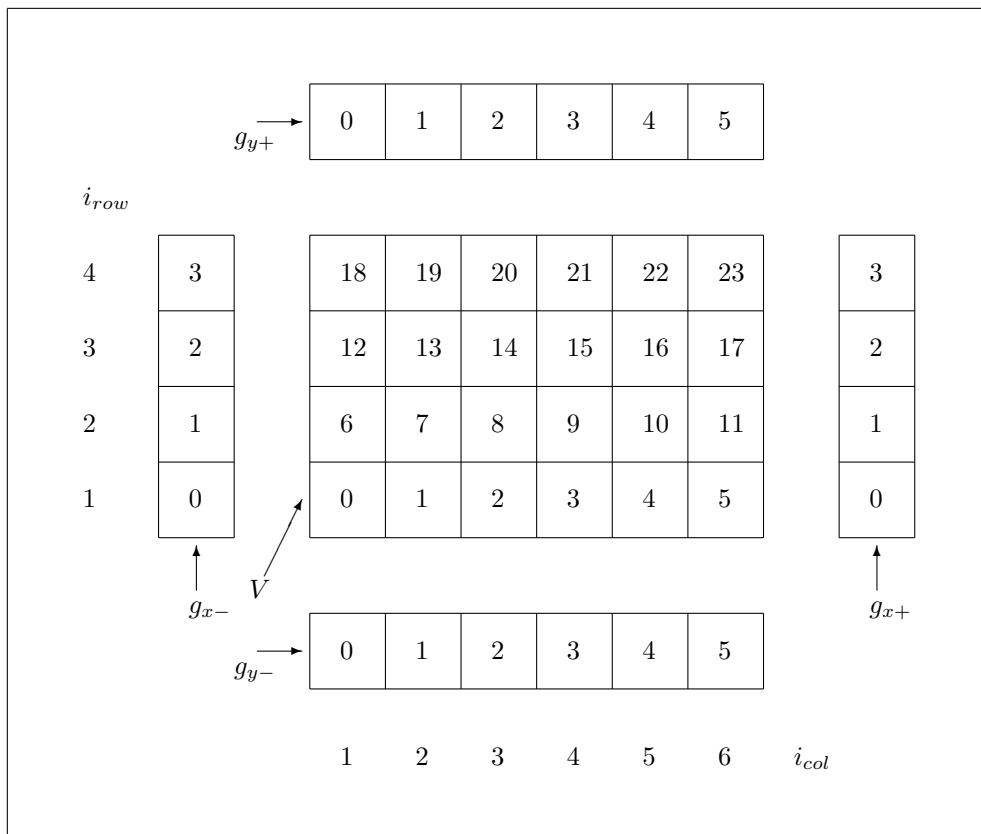
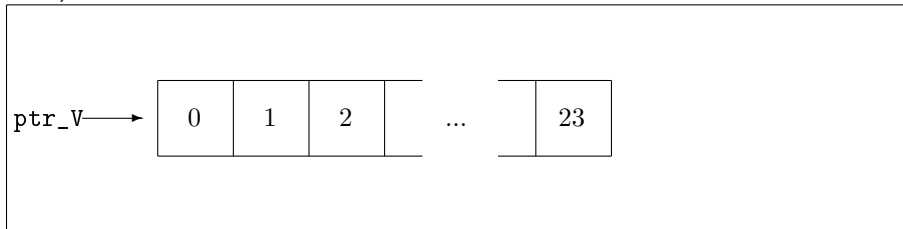


Figure 5: ANSI C - mesh XY type A

- 180 • $g_{x-} \equiv \text{double* ptr_gX_minus}$
- 181 • $g_{x+} \equiv \text{double* ptr_gX_plus}$
- 182 • $g_{y-} \equiv \text{double* ptr_gY_minus}$
- 183 • $g_{y+} \equiv \text{double* ptr_gY_plus}$
- 184 • $V \equiv \text{double* ptr_V}$
- 185 • `unsigned int size_row == 4`

- 186 • unsigned int size_col == 6
- 187 • unsigned int i_row == 1, 2, .., 4
- 188 • unsigned int i_col == 1,2, .., 6
- 189 • double h_x == 1.0 [mm]
- 190 • double h_y == 2.0 [mm]

191 The following picture describes analogous version of ptr_V mesh, which
 192 can be dynamically allocated on heap by pointer metod. The mesh is rep-
 193 resented by single block of memory. The numbers or rows and columns are
 194 also known, so each node can be also accessed by appropriate index (memory
 195 address).



196
 197 Each mesh point has its unique index (let's say icp - (index of central
 198 point)), which can be determined, if we know indices of row and column (i_row,
 199 i_col).

$$icp == (i_row - 1) * size_col + i_col - 1 \quad (9.1)$$

200 For example for each point of a mesh indices of row and column have val-
 201 ues:

$$\begin{aligned} i_row &== 1, 2, .. , size_row \\ i_col &== 1, 2, .. , size_col \end{aligned} \quad (9.2)$$

202 **10 Example of B-type mesh in ANSI C**

203 Example of B- type mesh in ANSI C program. The mesh is analogous to A -
204 type mesh. There are no electric field gradients on mesh borders.

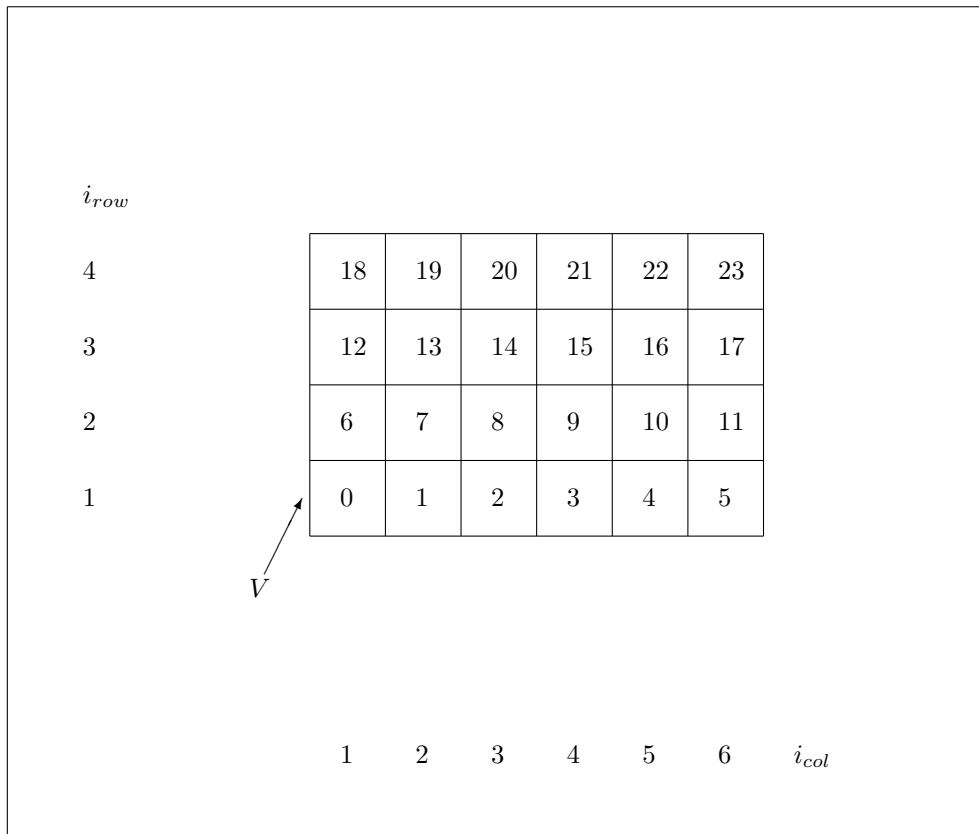


Figure 6: ANSI C - mesh XY type B

- 205 • $V \equiv \text{double* ptr}_V$
- 206 • `unsigned int size_row == 4`
- 207 • `unsigned int size_col == 6`
- 208 • `unsigned int i_row == 1, 2, ..., 4`
- 209 • `unsigned int i_col == 1,2, ..., 6`
- 210 • `double h_x == 1.0 [mm]`
- 211 • `double h_y == 2.0 [mm]`

212 **11 Example of C-type mesh in ANSI C**

213 Example of C- type mesh in ANSI C program. The mesh is analogous to A -
 214 type mesh. Just mesh mesh step $h_x = h_y = h$.

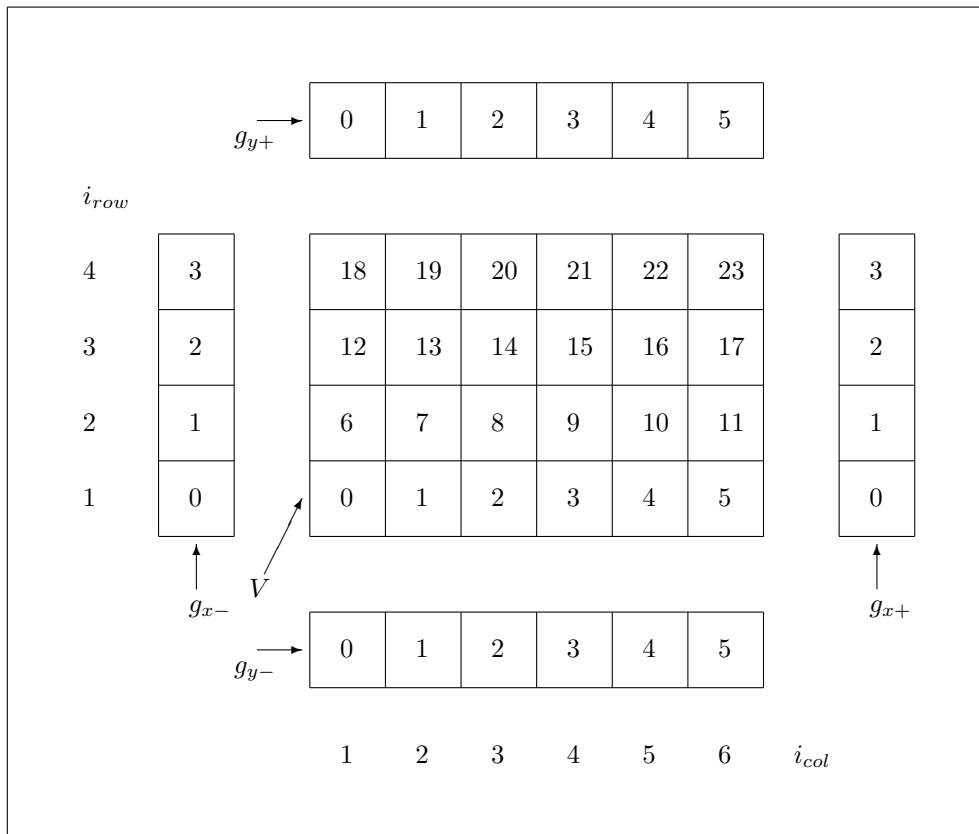


Figure 7: ANSI C - mesh XY type C

- 215 • $g_{x-} \equiv \text{double* ptr_gX_minus}$
- 216 • $g_{x+} \equiv \text{double* ptr_gX_plus}$
- 217 • $g_{y-} \equiv \text{double* ptr_gY_minus}$
- 218 • $g_{y+} \equiv \text{double* ptr_gY_plus}$
- 219 • $V \equiv \text{double* ptr_V}$
- 220 • `unsigned int size_row == 4`
- 221 • `unsigned int size_col == 6`
- 222 • `unsigned int i_row == 1, 2, .., 4`

```
223 • unsigned int i_col == 1,2, .., 6
224 • double h == 1.0 [mm]
```

225 **12 Example of D-type mesh in ANSI C**

226 Example of D- type mesh in ANSI C program. The mesh is analogous to B -
227 type mesh. Just $h_x = h_y = h$.

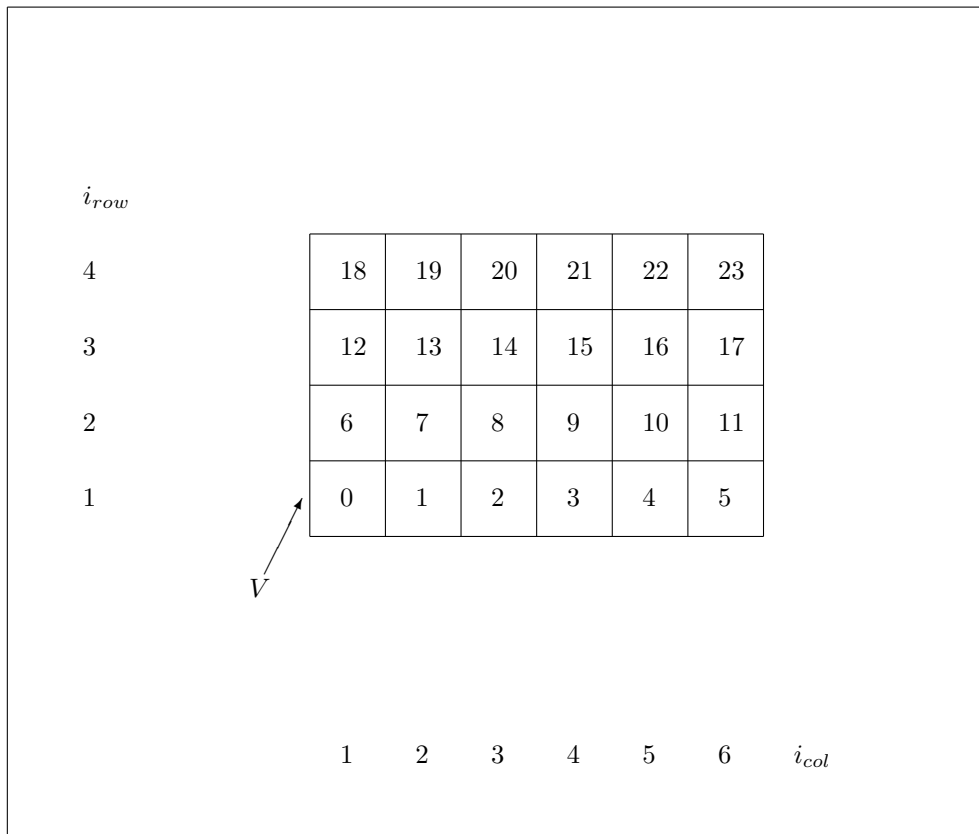


Figure 8: ANSI C - mesh XY type D

- 228 • $V \equiv \text{double* ptr}_V$
- 229 • `unsigned int size_row == 4`
- 230 • `unsigned int size_col == 6`
- 231 • `unsigned int i_row == 1, 2, ..., 4`
- 232 • `unsigned int i_col == 1,2, ..., 6`
- 233 • `double h == 1.0 [mm]`

234 **13 Relaxation formula for node P1**

235 **13.1 Node description**

236 Left, bottom corner of mesh XY.

237 **13.2 Calculation of relaxation formula**

238 Laplace equation at node P_1

$$\nabla^2 (V_{(x,y)})_{P_1} = 0 \quad (13.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_1} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_1} = 0 \quad (13.2)$$

239 Approximation of partial derivatives of $V_{(x,y)}$ at node P_1

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_1} \approx \frac{\frac{V_2 - V_1}{h_x} - \frac{V_1 - V_{1y-}}{h_x}}{h_x} = \frac{V_2 - V_1}{h_x^2} - \frac{g_{1x-}}{h_x} \quad (13.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_1} \approx \frac{\frac{V_4 - V_1}{h_y} - \frac{V_1 - V_{1y-}}{h_y}}{h_y} = \frac{V_4 - V_1}{h_y^2} - \frac{g_{1y-}}{h_y} \quad (13.4)$$

240 Let us substitute approximations to Laplace equation.

$$\frac{V_2 - V_1}{h_x^2} - \frac{g_{1x-}}{h_x} + \frac{V_4 - V_1}{h_y^2} - \frac{g_{1y-}}{h_y} = 0 \quad (13.5)$$

241 Let us find V_1

$$V_1 = ? \quad (13.6)$$

$$\frac{V_2 - V_1}{h_x^2} + \frac{V_4 - V_1}{h_y^2} = \frac{g_{1x-}}{h_x} + \frac{g_{1y-}}{h_y} \quad (13.7)$$

242 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (13.8)$$

243 We obtain

$$V_2 h_y^2 - V_1 h_y^2 + V_4 h_x^2 - V_1 h_x^2 = g_{1x-} h_x h_y^2 + g_{1y-} h_x^2 h_y \quad (13.9)$$

$$V_1 (h_x^2 + h_y^2) = V_2 h_y^2 + V_4 h_x^2 - g_{1x-} h_x h_y^2 - g_{1y-} h_x^2 h_y \quad (13.10)$$

244 **13.3 Final forms of relaxation formula**

245 **13.3.1 xyLV_RELAX5_P1_A**

$$\begin{aligned} & h_x \neq h_y \\ & g_{1x-}, g_{1y-} \neq 0 \\ 246 \quad V_1 &= \frac{V_2 h_y^2 + V_4 h_x^2 - g_{1x-} h_x h_y^2 - g_{1y-} h_x^2 h_y}{h_x^2 + h_y^2} \end{aligned} \quad (13.11)$$

247 **13.3.2 xyLV_RELAX5_P1_B**

$$\begin{aligned} & h_x \neq h_y \\ & g_{1x-}, g_{1y-} = 0 \\ V_1 &= \frac{V_2 h_y^2 + V_4 h_x^2}{h_x^2 + h_y^2} \end{aligned} \quad (13.12)$$

248 **13.3.3 xyLV_RELAX5_P1_C**

$$\begin{aligned} & h_x = h_y = h \\ & g_{1x-}, g_{1y-} \neq 0 \\ V_1 &= \frac{V_2 + V_4 - g_{1x-} h - g_{1y-} h}{2} \end{aligned} \quad (13.13)$$

249 **13.3.4 xyLV_RELAX5_P1_D**

$$\begin{aligned} & h_x = h_y = h \\ & g_{1x-}, g_{1y-} = 0 \\ V_1 &= \frac{V_2 + V_4}{2} \end{aligned} \quad (13.14)$$

250 **14 Relaxation formula for node P2**

251 **14.1 Node description**

252 Bottom edge of mesh XY.

253 **14.2 Calculation of relaxation formula**

254 Laplace equation at node P_2

$$\nabla^2 (V_{(x,y)})_{P_2} = 0 \quad (14.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_2} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_2} = 0 \quad (14.2)$$

255 Approximation of partial derivatives of $V_{(x,y)}$ at node P_2

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_2} \approx \frac{\frac{V_3 - V_2}{h_x} - \frac{V_2 - V_1}{h_x}}{h_x} = \frac{V_1 + V_3 - 2V_2}{h_x^2} \quad (14.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_2} \approx \frac{\frac{V_5 - V_2}{h_y} - \frac{V_2 - V_{2y-}}{h_y}}{h_y} = \frac{V_5 - V_2}{h_y^2} - \frac{g_{2y-}}{h_y} \quad (14.4)$$

256 Let us substitute approximations to Laplace equation.

$$\frac{V_1 + V_3 - 2V_2}{h_x^2} + \frac{V_5 - V_2}{h_y^2} - \frac{g_{2y-}}{h_y} = 0 \quad (14.5)$$

257 Let us find V_2

$$V_2 = ? \quad (14.6)$$

$$\frac{V_1 + V_3 - 2V_2}{h_x^2} + \frac{V_5 - V_2}{h_y^2} = \frac{g_{2y-}}{h_y} \quad (14.7)$$

258 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (14.8)$$

259 We obtain

$$V_1 h_y^2 + V_3 h_y^2 - 2V_2 h_y^2 + V_5 h_x^2 = g_{2y-} h_x^2 h_y \quad (14.9)$$

$$V_2 (h_x^2 + h_y^2) = (V_1 + V_3) h_y^2 + V_5 h_x^2 - g_{2y-} h_x^2 h_y \quad (14.10)$$

260 **14.3 Final forms of relaxation formula**

261 **14.3.1 xyLV_RELAX5_P2_A**

$$\begin{aligned} h_x &\neq h_y \\ g_{2y-} &\neq 0 \\ V_2 &= \frac{(V_1 + V_3) h_y^2 + V_5 h_x^2 - g_{2y-} h_x^2 h_y}{h_x^2 + h_y^2} \end{aligned} \quad (14.11)$$

262 **14.3.2 xyLV_RELAX5_P2_B**

$$\begin{aligned} h_x &\neq h_y \\ g_{2y-} &= 0 \\ V_2 &= \frac{(V_1 + V_3) h_y^2 + V_5 h_x^2}{h_x^2 + h_y^2} \end{aligned} \quad (14.12)$$

263 **14.3.3 xyLV_RELAX5_P2_C**

$$\begin{aligned} h_x &= h_y = h \\ g_{2y-} &\neq 0 \\ V_2 &= \frac{V_1 + V_3 + V_5 - g_{2y-} h}{3} \end{aligned} \quad (14.13)$$

264 **14.3.4 xyLV_RELAX5_P2_D**

$$\begin{aligned} h_x &= h_y = h \\ g_{2y-} &= 0 \\ V_2 &= \frac{V_1 + V_3 + V_5}{3} \end{aligned} \quad (14.14)$$

265 **15 Relaxation formula for node P3**

266 **15.1 Node description**

267 Right, bottom corner of mesh XY.

268 **15.2 Calculation of relaxation formula**

269 Laplace equation at node P_3

$$\nabla^2 (V_{(x,y)})_{P_3} = 0 \quad (15.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_3} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_3} = 0 \quad (15.2)$$

270 Approximation of partial derivatives of $V_{(x,y)}$ at node P_3

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_3} \approx \frac{V_{3x+} - V_3 - \frac{V_3 - V_2}{h_x}}{h_x} = \frac{g_{3x+}}{h_x} + \frac{V_2 - V_3}{h_x^2} \quad (15.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_3} \approx \frac{\frac{V_6 - V_3}{h_y} - \frac{V_3 - V_{3y-}}{h_y}}{h_y} = \frac{V_6 - V_3}{h_y^2} - \frac{g_{3y-}}{h_y} \quad (15.4)$$

271 Let us substitute approximations to Laplace equation.

$$\frac{g_{3x+}}{h_x} + \frac{V_2 - V_3}{h_x^2} + \frac{V_6 - V_3}{h_y^2} - \frac{g_{3y-}}{h_y} = 0 \quad (15.5)$$

272 Let us find V_3

$$V_3 = ? \quad (15.6)$$

$$\frac{V_2 - V_3}{h_x^2} + \frac{V_6 - V_3}{h_y^2} = \frac{g_{3y-}}{h_y} - \frac{g_{3x+}}{h_x} \quad (15.7)$$

273 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (15.8)$$

274 We obtain

$$V_2 h_y^2 - V_3 h_y^2 + V_6 h_x^2 - V_3 h_x^2 = g_{3y-} h_x^2 h_y - g_{3x+} h_x h_y^2 \quad (15.9)$$

$$V_3 (h_x^2 + h_y^2) = V_2 h_y^2 + V_6 h_x^2 + g_{3x+} h_x h_y^2 - g_{3y-} h_x^2 h_y \quad (15.10)$$

275 **15.3 Final forms of relaxation formula**

276 **15.3.1 xyLV_RELAX5_P3_A**

$$\begin{aligned} & h_x \neq h_y \\ & g_{3x+}, g_{3y-} \neq 0 \\ V_3 &= \frac{V_2 h_y^2 + V_6 h_x^2 + g_{3x+} h_x h_y^2 - g_{3y-} h_x^2 h_y}{h_x^2 + h_y^2} \end{aligned} \quad (15.11)$$

277 **15.3.2 xyLV_RELAX5_P3_B**

$$\begin{aligned} & h_x \neq h_y \\ & g_{3x+}, g_{3y-} = 0 \\ V_3 &= \frac{V_2 h_y^2 + V_6 h_x^2}{h_x^2 + h_y^2} \end{aligned} \quad (15.12)$$

278 **15.3.3 xyLV_RELAX5_P3_C**

$$\begin{aligned} & h_x = h_y = h \\ & g_{3x+}, g_{3y-} \neq 0 \\ V_3 &= \frac{V_2 + V_6 + g_{3x+} h - g_{3y-} h}{2} \end{aligned} \quad (15.13)$$

279 **15.3.4 xyLV_RELAX5_P3_D**

$$\begin{aligned} & h_x = h_y = h \\ & g_{3x+}, g_{3y-} = 0 \\ V_3 &= \frac{V_2 + V_6}{2} \end{aligned} \quad (15.14)$$

280 **16 Relaxation formula for node P4**

281 **16.1 Node description**

282 Left edge of mesh XY.

283 **16.2 Calculation of relaxation formula**

284 Laplace equation at node P_4

$$\nabla^2 (V_{(x,y)})_{P_4} = 0 \quad (16.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_4} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_4} = 0 \quad (16.2)$$

285 Approximation of partial derivatives of $V_{(x,y)}$ at node P_4

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_4} \approx \frac{\frac{V_5 - V_4}{h_x} - \frac{V_4 - V_{4x-}}{h_x}}{h_x} = \frac{V_5 - V_4}{h_x^2} - \frac{g_{4x-}}{h_x} \quad (16.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_4} \approx \frac{\frac{V_7 - V_4}{h_y} - \frac{V_4 - V_1}{h_y}}{h_y} = \frac{V_1 + V_7 - 2V_4}{h_y^2} \quad (16.4)$$

286 Let us substitute approximations to Laplace equation.

$$\frac{V_5 - V_4}{h_x^2} - \frac{g_{4x-}}{h_x} + \frac{V_1 + V_7 - 2V_4}{h_y^2} = 0 \quad (16.5)$$

287 Let us find V_4

$$V_4 = ? \quad (16.6)$$

$$\frac{V_5 - V_4}{h_x^2} + \frac{V_1 + V_7 - 2V_4}{h_y^2} = \frac{g_{4x-}}{h_x} \quad (16.7)$$

288 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (16.8)$$

289 We obtain

$$V_5 h_y^2 - V_4 h_y^2 + V_1 h_x^2 + V_7 h_x^2 - 2V_4 h_x^2 = g_{4x-} h_x h_y^2 \quad (16.9)$$

$$V_4 (2h_x^2 + h_y^2) = (V_1 + V_7) h_x^2 + V_5 h_y^2 - g_{4x-} h_x h_y^2 \quad (16.10)$$

290 **16.3 Final forms of relaxation formula**

291 **16.3.1 xyLV_RELAX5_P4_A**

$$\begin{aligned} h_x &\neq h_y \\ g_{4x-} &\neq 0 \\ V_4 &= \frac{(V_1 + V_7) h_x^2 + V_5 h_y^2 - g_{4x-} h_x h_y^2}{2h_x^2 + h_y^2} \end{aligned} \quad (16.11)$$

292 **16.3.2 xyLV_RELAX5_P4_B**

$$\begin{aligned} h_x &\neq h_y \\ g_{4x-} &= 0 \\ V_2 &= \frac{(V_1 + V_7) h_x^2 + V_5 h_y^2}{2h_x^2 + h_y^2} \end{aligned} \quad (16.12)$$

293 **16.3.3 xyLV_RELAX5_P4_C**

$$\begin{aligned} h_x &= h_y = h \\ g_{4x-} &\neq 0 \\ V_4 &= \frac{V_1 + V_5 + V_7 - g_{4x-} h}{3} \end{aligned} \quad (16.13)$$

294 **16.3.4 xyLV_RELAX5_P4_D**

$$\begin{aligned} h_x &= h_y = h \\ g_{4x-} &= 0 \\ V_4 &= \frac{V_1 + V_5 + V_7}{3} \end{aligned} \quad (16.14)$$

295 **17 Relaxation formula for node P5**

296 **17.1 Node description**

297 Node inside a mesh XY.

298 **17.2 Calculation of relaxation formula**

299 Laplace equation at node P_5

$$\nabla^2 (V_{(x,y)})_{P_5} = 0 \quad (17.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_5} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_5} = 0 \quad (17.2)$$

300 Approximation of partial derivatives of $V_{(x,y)}$ at node P_5

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_5} \approx \frac{\frac{V_6 - V_5}{h_x} - \frac{V_5 - V_4}{h_x}}{h_x} = \frac{V_4 + V_6 - 2V_5}{h_x^2} \quad (17.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_5} \approx \frac{\frac{V_8 - V_5}{h_y} - \frac{V_5 - V_2}{h_y}}{h_y} = \frac{V_2 + V_8 - 2V_5}{h_y^2} \quad (17.4)$$

301 Let us substitute approximations to Laplace equation.

$$\frac{V_4 + V_6 - 2V_5}{h_x^2} + \frac{V_2 + V_8 - 2V_5}{h_y^2} = 0 \quad (17.5)$$

302 Let us find V_5

$$V_5 = ? \quad (17.6)$$

$$\frac{V_4 + V_6 - 2V_5}{h_x^2} + \frac{V_2 + V_8 - 2V_5}{h_y^2} = 0 \quad (17.7)$$

303 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (17.8)$$

304 We obtain

$$V_4 h_y^2 + V_6 h_y^2 - 2V_5 h_y^2 + V_2 h_x^2 + V_8 h_x^2 - 2V_5 h_x^2 = 0 \quad (17.9)$$

$$2V_5 (h_x^2 + h_y^2) = (V_2 + V_8) h_x^2 + (V_4 + V_6) h_y^2 \quad (17.10)$$

305 **17.3 Final forms of relaxation formula**

306 **17.3.1 xyLV_RELAX5_P5_A**

$$h_x \neq h_y$$

307 No gradients g inside mesh are considered.

$$V_5 = \frac{(V_2 + V_8) h_x^2 + (V_4 + V_6) h_y^2}{2(h_x^2 + h_y^2)} \quad (17.11)$$

308 **17.3.2 xyLV_RELAX5_P5_B**

$$h_x \neq h_y$$

309 Relaxation formula is the same as xyLV_RELAX5_P5_A

$$V_5 = \frac{(V_2 + V_8) h_x^2 + (V_4 + V_6) h_y^2}{2(h_x^2 + h_y^2)} \quad (17.12)$$

310 **17.3.3 xyLV_RELAX5_P5_C**

$$h_x = h_y = h$$

311 No gradients g inside mesh are considered.

312 The formula simplifies, so no g and h terms are necessary.

$$V_5 = \frac{V_2 + V_4 + V_6 + V_8}{4} \quad (17.13)$$

313 **17.3.4 xyLV_RELAX5_P5_D**

$$h_x = h_y = h$$

314 The formula also simplifies.

315

316 Relaxation formula is the same as xyLV_RELAX5_P5_C

$$V_5 = \frac{V_2 + V_4 + V_6 + V_8}{4} \quad (17.14)$$

317 **18 Relaxation formula for node P6**

318 **18.1 Node description**

319 Right edge of mesh XY.

320 **18.2 Calculation of relaxation formula**

321 Laplace equation at node P_6

$$\nabla^2 (V_{(x,y)})_{P_6} = 0 \quad (18.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_6} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_6} = 0 \quad (18.2)$$

322 Approximation of partial derivatives of $V_{(x,y)}$ at node P_6

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_6} \approx \frac{\frac{V_{6x+} - V_6}{h_x} - \frac{V_6 - V_5}{h_x}}{h_x} = \frac{g_{6x+}}{h_x} + \frac{V_5 - V_6}{h_x^2} \quad (18.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_6} \approx \frac{\frac{V_9 - V_6}{h_y} - \frac{V_6 - V_3}{h_y}}{h_y} = \frac{V_3 + V_9 - 2V_6}{h_y^2} \quad (18.4)$$

323 Let us substitute approximations to Laplace equation.

$$\frac{g_{6x+}}{h_x} + \frac{V_5 - V_6}{h_x^2} + \frac{V_3 + V_9 - 2V_6}{h_y^2} = 0 \quad (18.5)$$

324 Let us find V_6

$$V_6 = ? \quad (18.6)$$

$$\frac{V_5 - V_6}{h_x^2} + \frac{V_3 + V_9 - 2V_6}{h_y^2} = -\frac{g_{6x+}}{h_x} \quad (18.7)$$

325 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (18.8)$$

326 We obtain

$$V_5 h_y^2 - V_6 h_y^2 + V_3 h_x^2 + V_9 h_x^2 - 2V_6 h_x^2 = -g_{6x+} h_x h_y^2 \quad (18.9)$$

$$V_6 (2h_x^2 + h_y^2) = (V_3 + V_9) h_x^2 + V_5 h_y^2 + g_{6x+} h_x h_y^2 \quad (18.10)$$

327 **18.3 Final forms of relaxation formula**

328 **18.3.1 xyLV_RELAX5_P6_A**

$$\begin{aligned} h_x &\neq h_y \\ g_{6x+} &\neq 0 \\ V_6 &= \frac{(V_3 + V_9) h_x^2 + V_5 h_y^2 + g_{6x+} h_x h_y^2}{2h_x^2 + h_y^2} \end{aligned} \quad (18.11)$$

329 **18.3.2 xyLV_RELAX5_P6_B**

$$\begin{aligned} h_x &\neq h_y \\ g_{6x+} &= 0 \\ V_6 &= \frac{(V_3 + V_9) h_x^2 + V_5 h_y^2}{2h_x^2 + h_y^2} \end{aligned} \quad (18.12)$$

330 **18.3.3 xyLV_RELAX5_P6_C**

$$\begin{aligned} h_x &= h_y = h \\ g_{6x+} &\neq 0 \\ V_6 &= \frac{V_3 + V_5 + V_9 + g_{6x+} h}{3} \end{aligned} \quad (18.13)$$

331 **18.3.4 xyLV_RELAX5_P6_D**

$$\begin{aligned} h_x &= h_y = h \\ g_{6x+} &= 0 \\ V_6 &= \frac{V_3 + V_5 + V_9}{3} \end{aligned} \quad (18.14)$$

332 **19 Relaxation formula for node P7**

333 **19.1 Node description**

334 Left, upper corner of mesh XY.

335 **19.2 Calculation of relaxation formula**

336 Laplace equation at node P_7

$$\nabla^2 (V_{(x,y)})_{P_7} = 0 \quad (19.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_7} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_7} = 0 \quad (19.2)$$

337 Approximation of partial derivatives of $V_{(x,y)}$ at node P_7

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_7} \approx \frac{\frac{V_8 - V_7}{h_x} - \frac{V_7 - V_{7x-}}{h_x}}{h_x} = \frac{V_8 - V_7}{h_x^2} - \frac{g_{7x-}}{h_x} \quad (19.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_7} \approx \frac{\frac{V_{7y+} - V_7}{h_y} - \frac{V_7 - V_4}{h_y}}{h_y} = \frac{V_4 - V_7}{h_y^2} + \frac{g_{7y+}}{h_y} \quad (19.4)$$

338 Let us substitute approximations to Laplace equation.

$$\frac{V_8 - V_7}{h_x^2} - \frac{g_{7x-}}{h_x} + \frac{V_4 - V_7}{h_y^2} + \frac{g_{7y+}}{h_y} = 0 \quad (19.5)$$

339 Let us find V_7

$$V_7 = ? \quad (19.6)$$

$$\frac{V_8 - V_7}{h_x^2} + \frac{V_4 - V_7}{h_y^2} = \frac{g_{7x-}}{h_x} - \frac{g_{7y+}}{h_y} \quad (19.7)$$

340 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (19.8)$$

341 We obtain

$$V_8 h_y^2 - V_7 h_y^2 + V_4 h_x^2 - V_7 h_x^2 = g_{7x-} h_x h_y^2 - g_{7y+} h_x^2 h_y \quad (19.9)$$

$$V_7 (h_x^2 + h_y^2) = V_4 h_x^2 + V_8 h_y^2 - g_{7x-} h_x h_y^2 - g_{7y+} h_x^2 h_y \quad (19.10)$$

342 **19.3 Final forms of relaxation formula**

343 **19.3.1 xyLV_RELAX5_P7_A**

$$\begin{aligned} h_x &\neq h_y \\ g_{7x-}, g_{7y+} &\neq 0 \\ V_7 &= \frac{V_4 h_x^2 + V_8 h_y^2 - g_{7x-} h_x h_y^2 + g_{7y+} h_x^2 h_y}{(h_x^2 + h_y^2)} \end{aligned} \quad (19.11)$$

344 **19.3.2 xyLV_RELAX5_P7_B**

$$\begin{aligned} h_x &\neq h_y \\ g_{7x-}, g_{7y+} &= 0 \\ V_7 &= \frac{V_4 h_x^2 + V_8 h_y^2}{h_x^2 + h_y^2} \end{aligned} \quad (19.12)$$

345 **19.3.3 xyLV_RELAX5_P7_C**

$$\begin{aligned} h_x &= h_y = h \\ g_{7x-}, g_{7y+} &\neq 0 \\ V_7 &= \frac{V_4 + V_8 - g_{7x-} h + g_{7y+} h}{2} \end{aligned} \quad (19.13)$$

346 **19.3.4 xyLV_RELAX5_P7_D**

$$\begin{aligned} h_x &= h_y = h \\ g_{7x-}, g_{7y+} &= 0 \\ V_7 &= \frac{V_4 + V_8}{2} \end{aligned} \quad (19.14)$$

347 **20 Relaxation formula for node P8**

348 **20.1 Node description**

349 Upper edge of mesh XY.

350 **20.2 Calculation of relaxation formula**

351 Laplace equation at node P_8

$$\nabla^2 (V_{(x,y)})_{P_8} = 0 \quad (20.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_8} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_8} = 0 \quad (20.2)$$

352 Approximation of partial derivatives of $V_{(x,y)}$ at node P_8

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_8} \approx \frac{\frac{V_9 - V_8}{h_x} - \frac{V_8 - V_7}{h_x}}{h_x} = \frac{V_7 + V_9 - 2V_8}{h_x^2} \quad (20.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_8} \approx \frac{\frac{V_{8y+} - V_8}{h_y} - \frac{V_8 - V_5}{h_y}}{h_y} = \frac{V_5 - V_8}{h_y^2} + \frac{g_{8y+}}{h_y} \quad (20.4)$$

353 Let us substitute approximations to Laplace equation.

$$\frac{V_7 + V_9 - 2V_8}{h_x^2} + \frac{V_5 - V_8}{h_y^2} + \frac{g_{8y+}}{h_y} = 0 \quad (20.5)$$

354 Let us find V_8

$$V_8 = ? \quad (20.6)$$

$$\frac{V_7 + V_9 - 2V_8}{h_x^2} + \frac{V_5 - V_8}{h_y^2} = -\frac{g_{8y+}}{h_y} \quad (20.7)$$

355 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (20.8)$$

356 We obtain

$$V_7 h_y^2 + V_9 h_y^2 - 2V_8 h_y^2 + V_5 h_x^2 - V_8 h_x^2 = -g_{8y+} h_x^2 h_y \quad (20.9)$$

$$V_8 (h_x^2 + 2h_y^2) = (V_7 + V_9) h_y^2 + V_5 h_x^2 + g_{8y+} h_x^2 h_y \quad (20.10)$$

357 **20.3 Final forms of relaxation formula**

358 **20.3.1 xyLV_RELAX5_P8_A**

$$\begin{aligned} h_x &\neq h_y \\ g_{8y+} &\neq 0 \\ V_8 &= \frac{V_5 h_x^2 + (V_7 + V_9) h_y^2 + g_{8y+} h_x^2 h_y}{h_x^2 + 2h_y^2} \end{aligned} \quad (20.11)$$

359 **20.3.2 xyLV_RELAX5_P8_B**

$$\begin{aligned} h_x &\neq h_y \\ g_{8y+} &= 0 \\ V_8 &= \frac{V_5 h_x^2 + (V_7 + V_9) h_y^2}{h_x^2 + 2h_y^2} \end{aligned} \quad (20.12)$$

360 **20.3.3 xyLV_RELAX5_P8_C**

$$\begin{aligned} h_x &= h_y = h \\ g_{8y+} &\neq 0 \\ V_8 &= \frac{V_5 + V_7 + V_9 + g_{8y+} h}{3} \end{aligned} \quad (20.13)$$

361 **20.3.4 xyLV_RELAX5_P8_D**

$$\begin{aligned} h_x &= h_y = h \\ g_{8y+} &= 0 \\ V_8 &= \frac{V_5 + V_7 + V_9}{3} \end{aligned} \quad (20.14)$$

362 **21 Relaxation formula for node P9**

363 **21.1 Node description**

364 Right, upper corner of mesh XY.

365 **21.2 Calculation of relaxation formula**

366 Laplace equation at node P_9

$$\nabla^2 (V_{(x,y)})_{P_9} = 0 \quad (21.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_9} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_9} = 0 \quad (21.2)$$

367 Approximation of partial derivatives of $V_{(x,y)}$ at node P_9

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_9} \approx \frac{\frac{V_{9x+} - V_9}{h_x} - \frac{V_9 - V_8}{h_x}}{h_x} = \frac{V_8 - V_9}{h_x^2} + \frac{g_{9x+}}{h_x} \quad (21.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_9} \approx \frac{\frac{V_{9y+} - V_9}{h_y} - \frac{V_9 - V_6}{h_y}}{h_y} = \frac{V_6 - V_9}{h_y^2} + \frac{g_{9y+}}{h_y} \quad (21.4)$$

368 Let us substitute approximations to Laplace equation.

$$\frac{V_8 - V_9}{h_x^2} + \frac{g_{9x+}}{h_x} + \frac{V_6 - V_9}{h_y^2} + \frac{g_{9y+}}{h_y} = 0 \quad (21.5)$$

369 Let us find V_9

$$V_9 =? \quad (21.6)$$

$$\frac{V_8 - V_9}{h_x^2} + \frac{V_8 - V_9}{h_x^2} = -\frac{g_{9x+}}{h_x} - \frac{g_{9y+}}{h_y} \quad (21.7)$$

370 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (21.8)$$

371 We obtain

$$V_8 h_y^2 - V_9 h_y^2 + V_6 h_x^2 - V_9 h_x^2 = -g_{9x+} h_x h_y^2 - g_{9y+} h_x^2 h_y \quad (21.9)$$

$$V_9 (h_x^2 + h_y^2) = V_6 h_x^2 + V_8 h_y^2 + g_{9x+} h_x h_y^2 + g_{9y+} h_x^2 h_y \quad (21.10)$$

372 **21.3 Final forms of relaxation formula**

373 **21.3.1 xyLV_RELAX5_P9_A**

$$\begin{aligned} & h_x \neq h_y \\ & g_{9x+}, g_{9y+} \neq 0 \\ V_9 &= \frac{V_6 h_x^2 + V_8 h_y^2 + g_{9x+} h_x h_y^2 + g_{9y+} h_x^2 h_y}{h_x^2 + h_y^2} \end{aligned} \quad (21.11)$$

374 **21.3.2 xyLV_RELAX5_P9_B**

$$\begin{aligned} & h_x \neq h_y \\ & g_{9x+}, g_{9y+} = 0 \\ V_9 &= \frac{V_6 h_x^2 + V_8 h_y^2}{h_x^2 + h_y^2} \end{aligned} \quad (21.12)$$

375 **21.3.3 xyLV_RELAX5_P9_C**

$$\begin{aligned} & h_x = h_y = h \\ & g_{9x+}, g_{9y+} \neq 0 \\ V_9 &= \frac{V_6 + V_8 + g_{9x+} h + g_{9y+} h}{2} \end{aligned} \quad (21.13)$$

376 **21.3.4 xyLV_RELAX5_P9_D**

$$\begin{aligned} & h_x = h_y = h \\ & g_{9x+}, g_{9y+} = 0 \\ V_9 &= \frac{V_6 + V_8}{2} \end{aligned} \quad (21.14)$$

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