

1

Liebmann technical documentation



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3

Laplace equation 2D (ZR)

4

(Cylindrical coordinates).

5

relaxation scheme explained.

6

(5 - point star)

7

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118 **1 Liebmann technical documentation series**

- 119 1. Wyznaczanie rozkładu pola elektrostatycznego w próżni metodą relak-
120 sacyjną Liebmann. (Polish version / wersja polska)
- 121 2. Determination of electrostatic field distribution by using Liebmann relax-
122 ation method. (English version / wersja angielska)
- 123 3. Graphics. Mapping voltages to colours. (colormaps)
- 124 4. Laplace equation 2D (XY). (Cartesian coordinates). Relaxation scheme
125 explained. (5 - point star)
- 126 5. Laplace equation 2D (ZR) (Cylindrical coordinates). Relaxation scheme
127 explained. (5 - point star)
- 128 6. Liebmann source code. (ANSI C programming language)

129 **2 Versions of this document**

- 130 1. version 1 - 2023.11.03
- 131 2. version 2 - 2023.01.04
- 132 3. version 3 - 2024.02.02
- 133 4. version 4 - 2024.04.02
- 134 5. version 5 - 2024.05.18
- 135 6. version 6 - 2024.05.23
- 136 7. version 7 - 2024.05.24
- 137 8. version 8 - 2024.06.06 (complete $P_1..P_9$)
- 138 9. version 9 - 2024.06.09
- 139 10. version 10 - 2024.07.17
- 140 11. version 11 - 2024.07.18

141 **3 Solving Laplace equation using relaxation method**

142 I tried to solve Laplace equation using mainly information from Pierre Grivet's
143 book (Electron Optics) - [1].
144 There are few editions of this book (1965, 1972). Second edition (1972) con-
145 tains explanation of relaxation method (page 38).

146 More generalized approaches has been drafted by James R. Nagel - [2].
147 <https://my.ece.utah.edu/~ece6340/LECTURES/Feb1/> (visited 2023-03-01).

148

149 Taylor expansion in cylindrical coordinates has been found on the Internet:
150 [3].

151

152 There are also publications edited by Albert Septier: Focusing of Charged
153 Particles [4] and Applied Charged Particle Optics (part A). [5].

154 I have also found some ideas in publication of D W O Heddle: Electrostatic
155 Lens Systems [6] (especially using PC computers to solve electrostatic prob-
156 lems).

157 I have also found (brief) description of by - hand solving of Laplace equa-
158 tion by Bohdan Paszkowski - [7] (Polish edition). English translation of this book
159 also exists - [8].

160

161 I would like to thank many people, who helped me with this challenge. Espe-
162 cially prof. dr hab. Mieczysław Jałochowski (supervisor of my master's thesis),
163 who enabled me to use SIMION and MATLAB software while writing master's
164 thesis about electron optical systems at University of Maria Curie - Skłodowska
165 in Lublin in 2008. I would also thank to prof. Marcin Turek for fruitful discus-
166 sion about numerical methods. What is more, my colleague Bartosz in 2012
167 had explained me general problems with software efficiency. So he had also
168 contributed significantly to the idea of Liebmann software (especially using C
169 language).

170 **4 Explanation of symbols in calculations**

- 171 • P_i - i -th mesh node
- 172 • V_i - value of electrostatic potential at node P_i . Unit - [V]
- 173 • h - mesh step (for example h_x - mesh step in x direction). Unit - [mm]
- 174 • $g_{i+/-}$ - gradient in direction i (for example $g_{1z-} = \frac{V_1 - V_{1z-}}{h_z}$). Unit - $[\frac{V}{mm}]$
- 175 • i_{row} - index of row in mesh. Values of $i_{row} = 1, 2, \dots, \text{size_row}$
- 176 • i_{col} - index of column in mesh. Values of $i_{col} = 1, 2, \dots, \text{size_col}$
- 177 • p - in book: - [1] $r = ph_r$, so for off - axis point we have: $p = (i_{row} - 1)$

178 **5 Mesh ZR - type A (on axis)**

179 $h_z \neq h_r$

180 gradient V outside a mesh exists

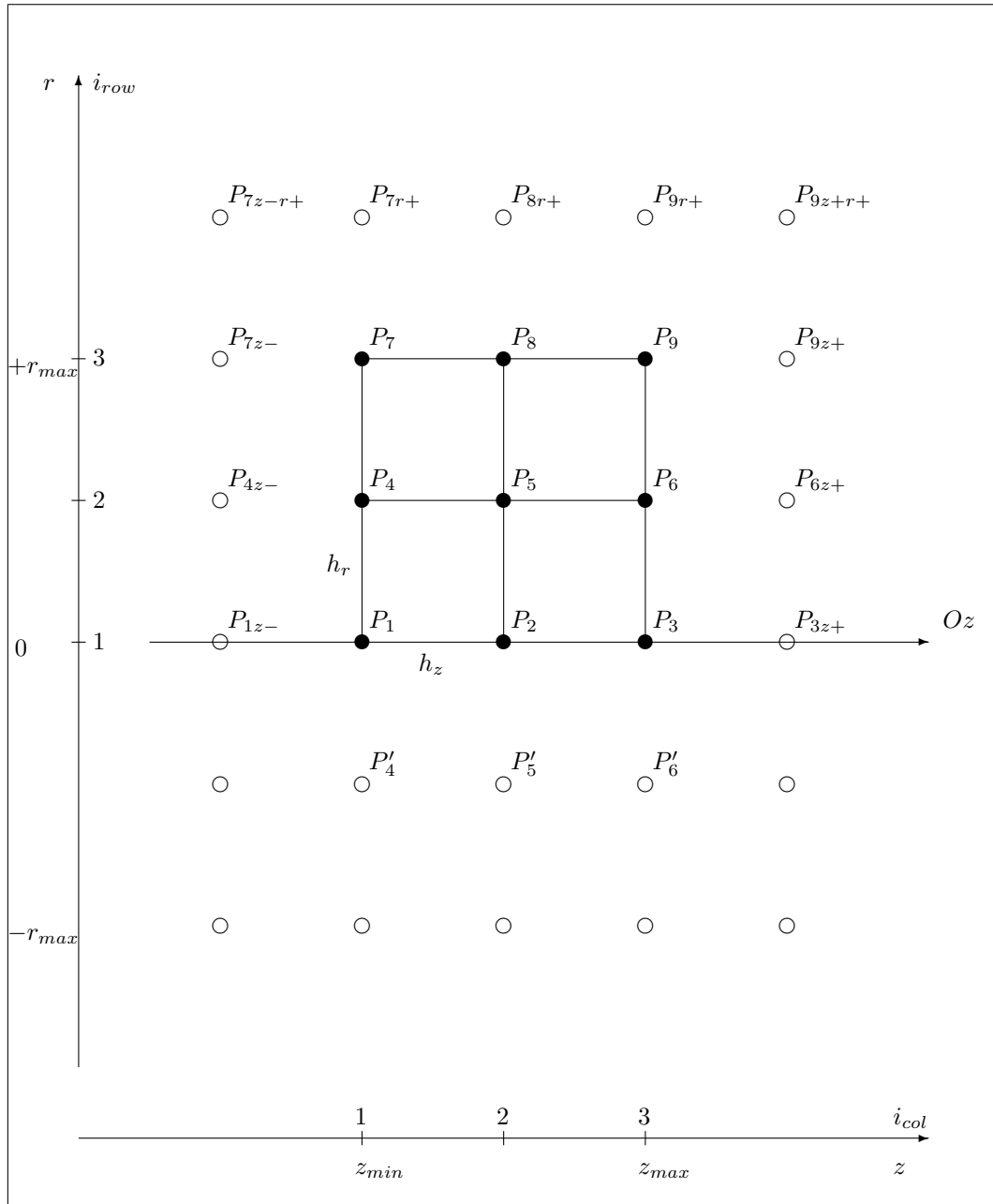


Figure 1: Mesh ZR type A

181 **6 Mesh ZR - type B (on axis)**

182 $h_z \neq h_r$

183 gradient V outside a mesh does not exist

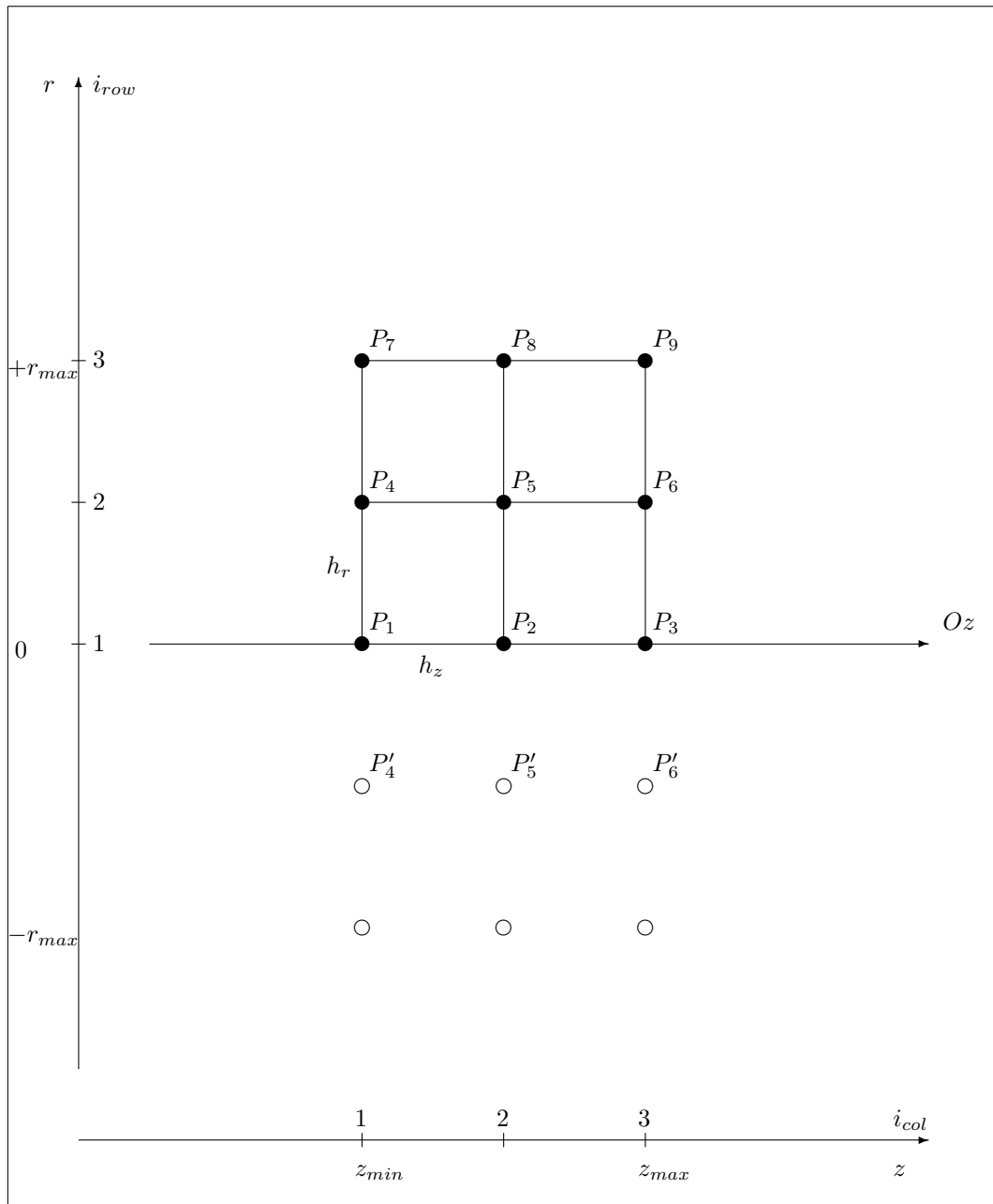


Figure 2: Mesh ZR type B

184 **7 Mesh ZR - type C (on axis)**

185 $h_z = h_r = h$

186 gradient V outside a mesh exists

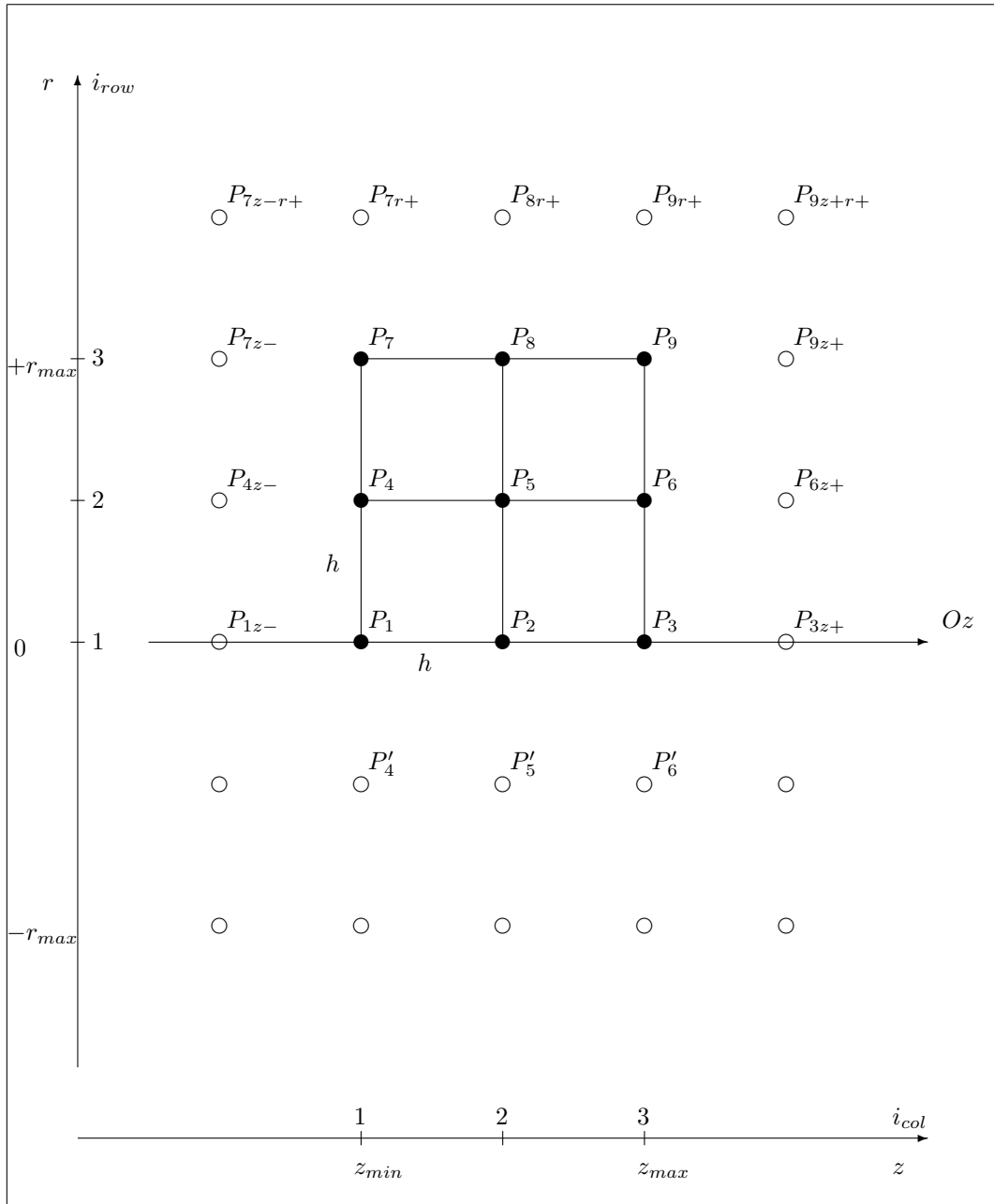


Figure 3: Mesh ZR type C

187 **8 Mesh ZR - type D (on axis)**

188 $h_z = h_r = h$

189 gradient V outside a mesh does not exist

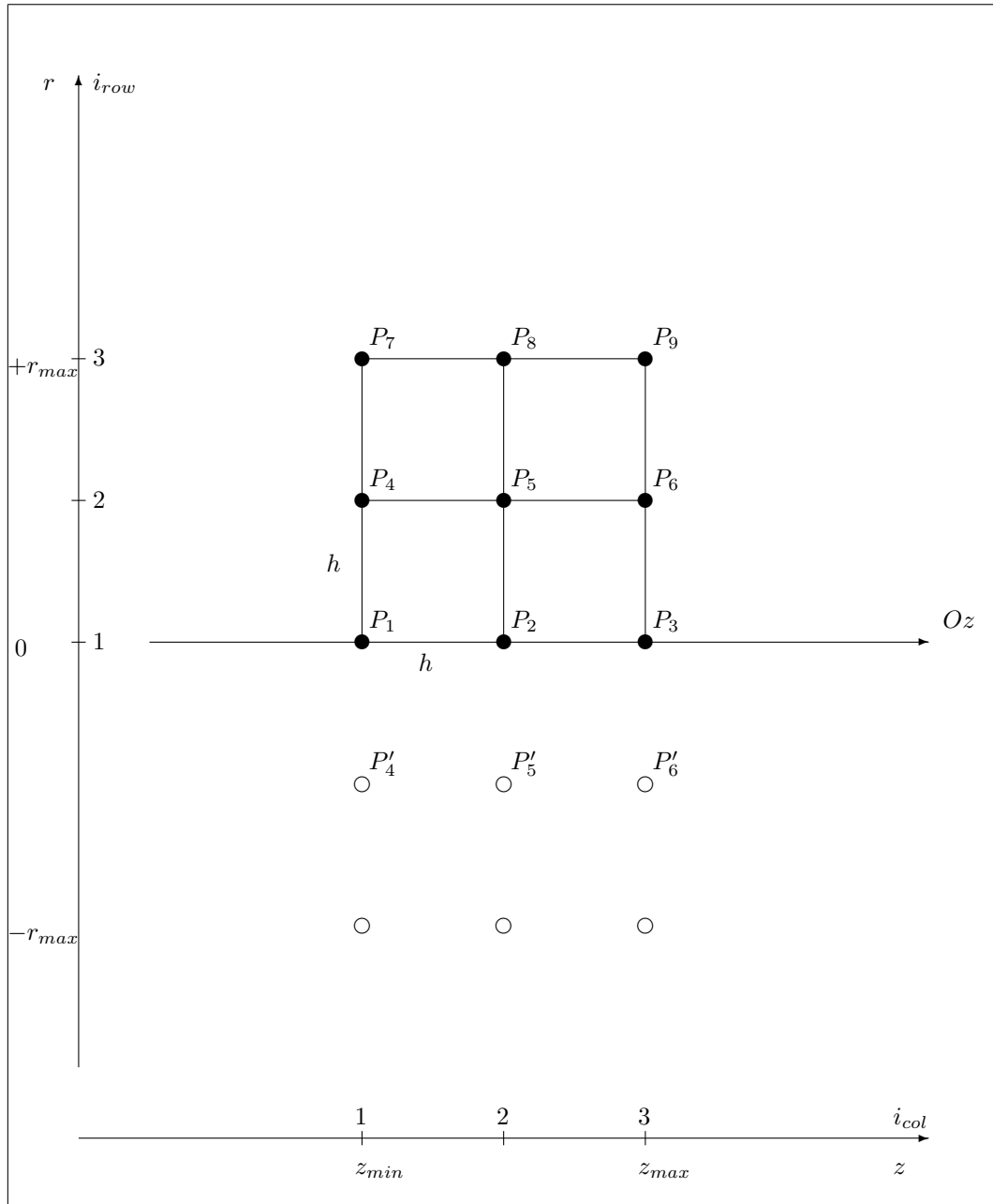


Figure 4: Mesh ZR type D

190 **9 Example of A-type mesh in ANSI C (on axis)**

191 Example of A- type mesh in ANSI C program. The mesh is represented by 2
 192 dimensional array of double precision numbers. Rows and columns in mesh
 193 are numbered from 1 (this was my choice) instead of default 0 (as usual in C
 194 language). This choice has pros and cons. It is easier to calculate mesh size
 195 (size_row * size_col). Access to each node can be also more intuitive, but logic
 196 in each library function must contain this shift between node ordering styles.

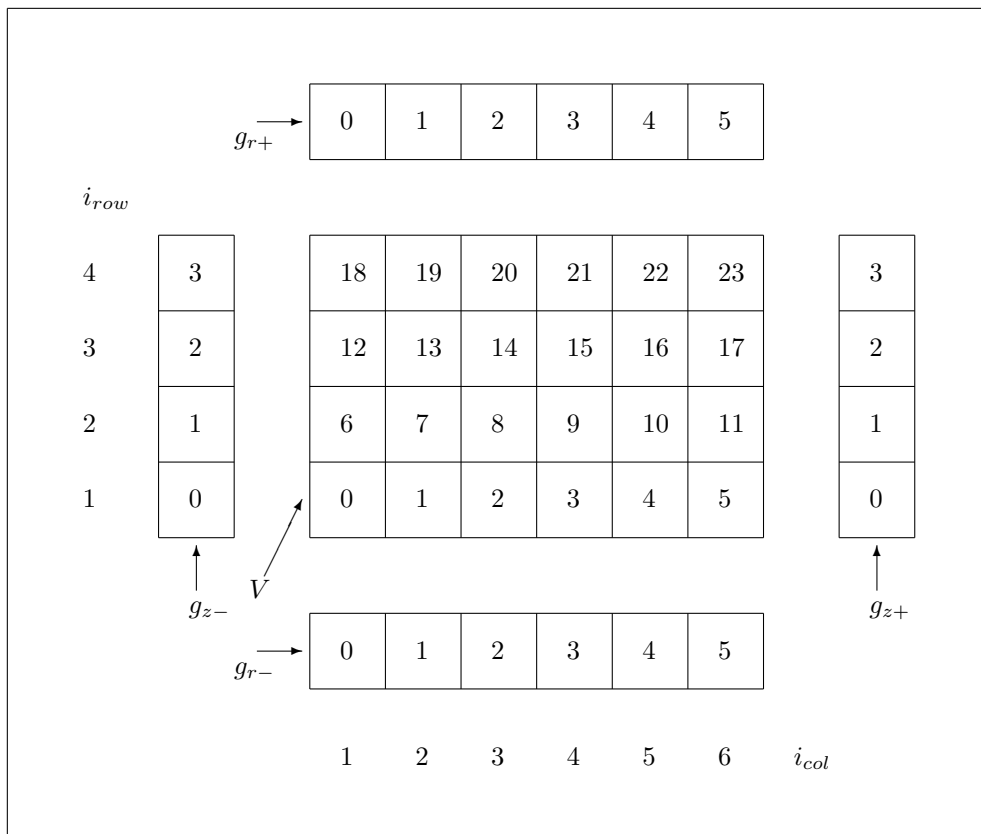


Figure 5: ANSI C - mesh XY type A

197 Note. This is more general example of „off-axis” mesh. If bottom edge of
 198 mesh lies on axis Oz , then gradient g_{r-} does not exist.

- 199 • $g_{z-} \equiv \text{double* ptr_gZ_minus}$
- 200 • $g_{z+} \equiv \text{double* ptr_gZ_plus}$
- 201 • $g_{r-} \equiv \text{double* ptr_gR_minus}$
- 202 • $g_{r+} \equiv \text{double* ptr_gR_plus}$

```

203 •  $V \equiv \text{double}^* \text{ptr}_V$ 
204 • unsigned int size_row == 4
205 • unsigned int size_col == 6
206 • unsigned int i_row == 1, 2, .., 4
207 • unsigned int i_col == 1,2, .., 6
208 • double h_z == 1.0 [mm]
209 • double h_r == 2.0 [mm]

```

210 The following picture describes analogous version of `ptr_V` mesh, which
 211 can be dynamically allocated on heap by pointer metod. The mesh is rep-
 212 resented by single block of memory. The numbers or rows and columns are
 213 also known, so each node can be also accessed by appropriate index (memory
 214 address).

215 The following picture describes analogous version of mesh, which can be
 216 easily dynamically allocated on heap by pointer metod. The mesh is repre-
 217 sented by single block of memory. The numbers or rows and columns are also
 218 known, so each node can be also accessed by appropriate index (memory ad-
 219 dress).

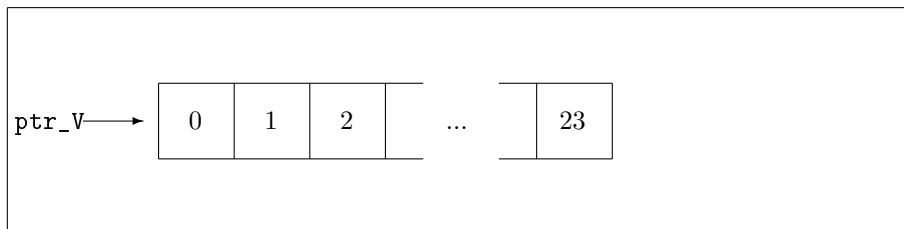


Figure 6: ANSI C - mesh ZR type D

220 Each mesh point has its unique index (let's say `icp` - (index of central
 221 point)), which can be determined, if we know indices of row and column (`i_row`,
 222 `i_col`).

$$\text{icp} == (\text{i_row} - 1) * \text{size_col} + \text{i_col} - 1 \quad (9.1)$$

223 For example for each point of a mesh indices of row and column have val-
 224 ues:

$$\begin{aligned} \text{i_row} &== 1, 2, \dots, \text{size_row} \\ \text{i_col} &== 1, 2, \dots, \text{size_col} \end{aligned} \quad (9.2)$$

225 Also for any relaxation formula for off - axis case the p symbol appears. This
226 symbol is connected with r cylindrical coordinate of given node:
227

$$r = ph_r \quad (9.3)$$

228 SO:

$$p == (i_row - 1) \quad (9.4)$$

229 **10 Example of B-type mesh in ANSI C (on axis)**

230 Example of B- type mesh in ANSI C program. The mesh is analogous to A -
231 type mesh. There are no electric field gradients on mesh borders.

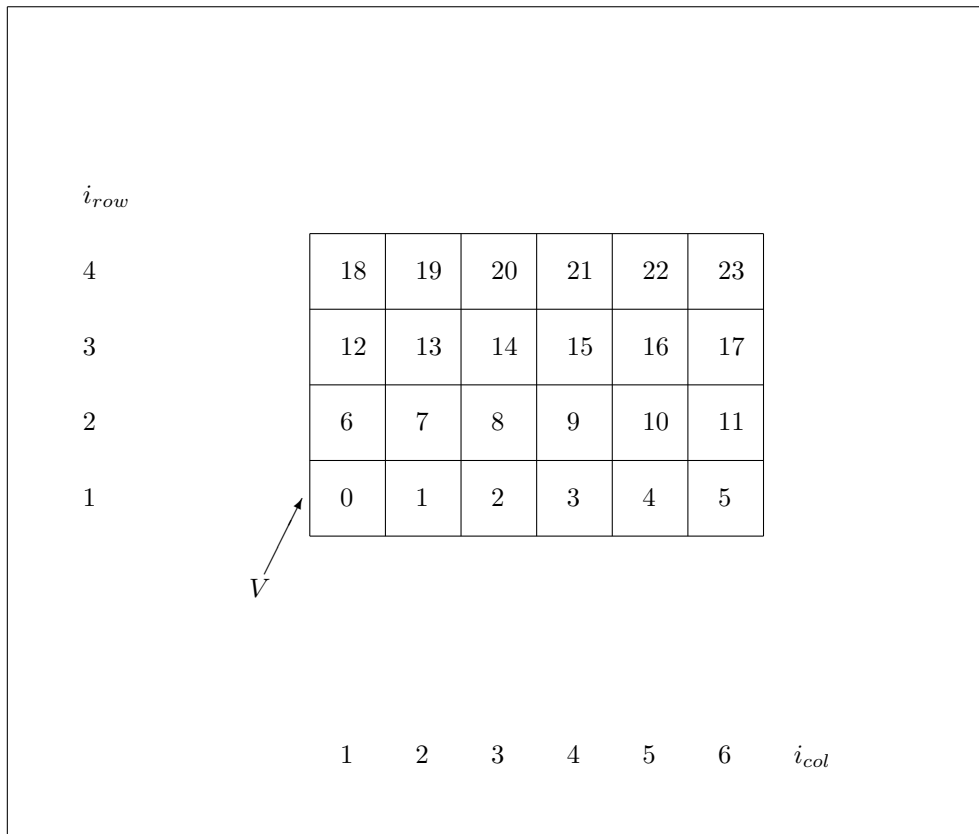


Figure 7: ANSI C - mesh XY type B

- 232 • $V \equiv \text{double* ptr}_V$
- 233 • `unsigned int size_row == 4`
- 234 • `unsigned int size_col == 6`
- 235 • `unsigned int i_row == 1, 2, .., 4`
- 236 • `unsigned int i_col == 1,2, .., 6`
- 237 • `double h_z == 1.0 [mm]`
- 238 • `double h_r == 2.0 [mm]`

239 **11 Example of C-type mesh in ANSI C (on axis)**

240 Example of C- type mesh in ANSI C program. The mesh is analogous to A -
 241 type mesh. Just mesh mesh step $h_x = h_y = h$.

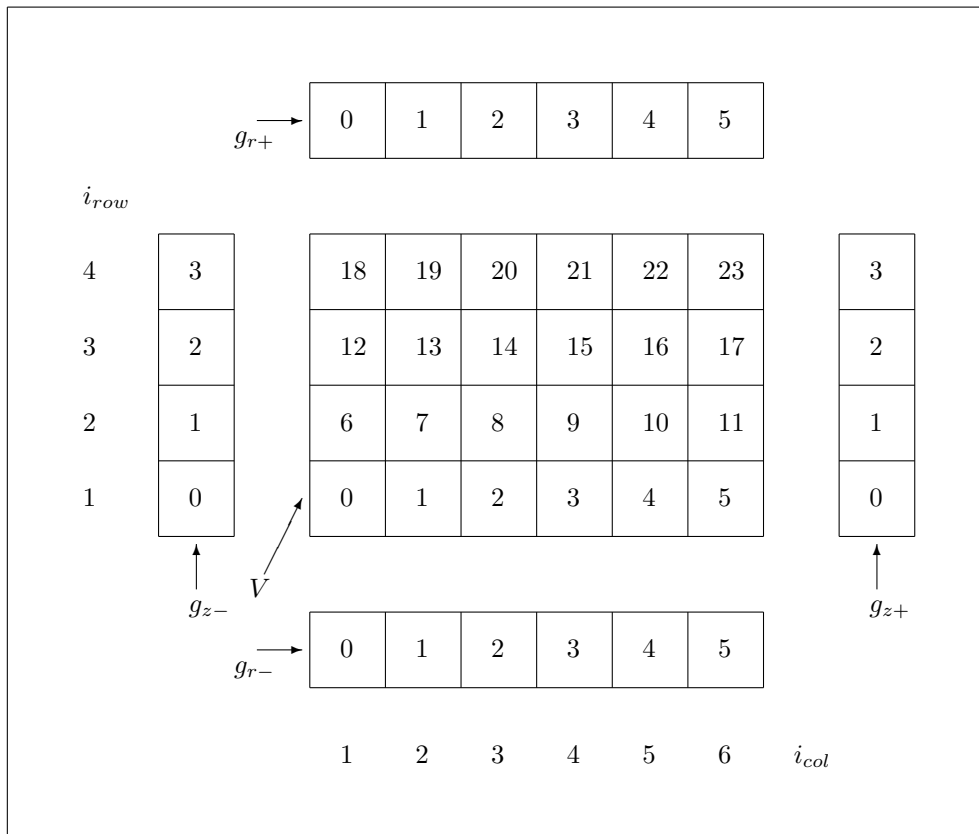


Figure 8: ANSI C - mesh XY type C

242 Note. This is more general example of „off-axis” mesh. If bottom egde of
 243 mesh lies on axis Oz , then gradient g_{r-} does not exist.

- 244 • $g_{z-} \equiv \text{double* ptr_gZ_minus}$
- 245 • $g_{z+} \equiv \text{double* ptr_gZ_plus}$
- 246 • $g_{r-} \equiv \text{double* ptr_gR_minus}$
- 247 • $g_{r+} \equiv \text{double* ptr_gR_plus}$
- 248 • $V \equiv \text{double* ptr_V}$
- 249 • `unsigned int size_row == 4`


```
250 • unsigned int size_col == 6
251 • unsigned int i_row == 1, 2, .., 4
252 • unsigned int i_col == 1,2, .., 6
253 • double h == 1.0 [mm]
```

254 **12 Example of D-type mesh in ANSI C (on axis)**

255 Example of D- type mesh in ANSI C program. The mesh is analogous to B -
256 type mesh. Just $h_x = h_y = h$.

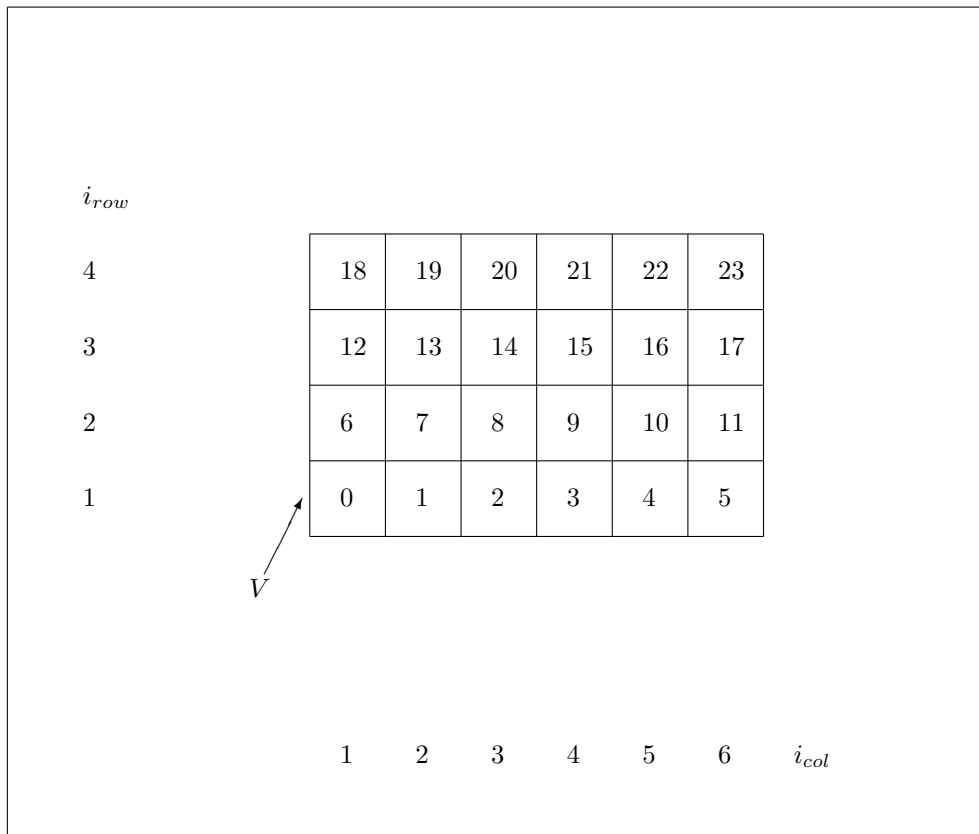


Figure 9: ANSI C - mesh ZR type D

- 257 • $V \equiv \text{double* ptr}_V$
- 258 • `unsigned int size_row == 4`
- 259 • `unsigned int size_col == 6`
- 260 • `unsigned int i_row == 1, 2, .., 4`
- 261 • `unsigned int i_col == 1,2, .., 6`
- 262 • `double h == 1.0 [mm]`

263 **13 Partial derivatives on Oz axis**

264 **13.1 Personal note**

265 This is my personal interpretation. I cannot guarantee correctness of this ap-
266 proach

267 **13.2 Nodes numbering (on axis Oz)**

268 We will try to work with P_2 point (determine approximations of aprtial derivatives
269 for point P_2 , which lies on axis Oz). Nodes numbering on axis Oz differs from
270 numbering convention in Pierre Grivet's book.

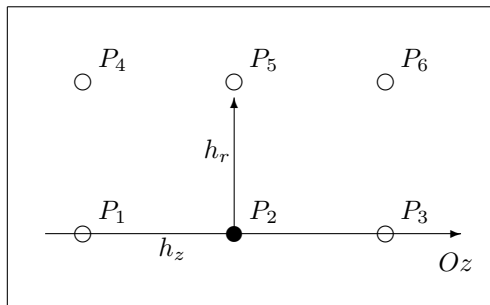


Figure 10: Nodes on axis Oz

271 Point P_2 is situated on Oz axis. It has 2 neighbours on axis Oz : P_1 and P_3 .
272 Node P_5 lies above P_2 node. The mesh step in r direction is h_r . The mesh
273 step in z direction is h_z .

274 **13.3 Taylor expansion of $V_{(z,r)}$ function in cylindrical coordinates**

275 Taylor expansion of $V_{(z,r)}$ function in cylindrical coordinates [3]:

$$\begin{aligned}
 V_{(z,r)} = V_{(z_0,r_0)} + \left(\frac{\partial V}{\partial z} \right)_{(z_0,r_0)} (z - z_0) + \\
 \left(\frac{\partial V}{\partial r} \right)_{(z_0,r_0)} (r - r_0) + \\
 \frac{1}{2!} \left(\frac{\partial^2 V}{\partial z^2} \right)_{(z_0,r_0)} (z - z_0)^2 + \dots
 \end{aligned} \tag{13.1}$$

276 **13.4 Laplace operator in rotationally symmetrical systems**

277 Laplace operator in cylindrical coordinates (cylindrical symmetry) consist of 3
278 elements [1] (on page 42):

$$\nabla^2 (V_{(z,r)}) = \left(\frac{\partial^2 V}{\partial r^2} \right) + \frac{1}{r} \left(\frac{\partial V}{\partial r} \right) + \left(\frac{\partial^2 V}{\partial z^2} \right) \quad (13.2)$$

279 In this chapter we will try to determine approximation of each term.

280 **13.5 Value of first partial derivative of V with respect to r on axis**
281 Oz

282 In cylindrically symmetrical field first partial derivative of V (by r) on axis Oz
283 equals zero (because $V_{(+dr)} = V_{(-dr)}$)

$$\left(\frac{\partial V}{\partial r} \right)_{(z,r=0)} = 0 \quad (13.3)$$

284 **13.6 Value of second partial derivative of V with respect to r on**
285 **axis Oz**

286 In this subchapter we will try to determine the first term of equation 13.2

287 In our case there is node P_2 on axis Oz . The nearest neighbour of P_2 is
288 node P_5 , which lies „over Oz axis“. The distance between P_2 and P_5 is h_r .
289 When we „walk away“ axis Oz in r direction (from point P_2 to point P_5), the
290 electric potential V_5 can be determined from truncated Taylor expansion 13.1
291 by expression:

$$V_5 \approx V_2 + \left(\frac{\partial V}{\partial r} \right)_{P_2} \cdot h_r + \frac{1}{2!} \left(\frac{\partial^2 V}{\partial r^2} \right)_{P_2} \cdot h_r^2 \quad (13.4)$$

292 We want to determine the second derivative:

$$\left(\frac{\partial^2 V}{\partial r^2} \right)_{P_2} = ? \quad (13.5)$$

293 We solve equation 13.4 (using relation 13.3).

$$\left(\frac{\partial^2 V}{\partial r^2} \right)_{P_2} \approx \frac{2!(V_5 - V_2)}{h_r^2} = \frac{2(V_5 - V_2)}{h_r^2} \quad (13.6)$$

294 This is final form of approximation of the second derivative of V with respect
295 to r on axis Oz . It will help us to determine Laplace operator in rotationally
296 symmetrical systems.

297 **13.7 Value of first partial derivative of V with respect to r divided**
 298 **by r on axis Oz**

299 We will try to determine the second term of relation 13.2 When we are on Oz
 300 axis, the second term has to be determined (because it aims to value $\frac{0}{0}$).

301 When we „ walk away” axis Oz in r direction, the electric potential $V_{(z_0,r)}$
 302 can be determined from truncated Taylor expansion by:

303

$$V_{(z_0,r)} \approx V_{(z_0,r_0)} + \left(\frac{\partial V}{\partial r} \right)_{(z_0,r_0)} (r - r_0) \quad (13.7)$$

304 On Oz axis $r_0 = 0$, so $(r_0 - r) = r$

305

306 Thus we have:

$$V_{(z_0,r)} \approx V_{(z_0,0)} + \left(\frac{\partial V}{\partial r} \right)_{(z_0,0)} \cdot r \quad (13.8)$$

307 Now let us differentiate (both sides) of such relation:

$$\left| \frac{\partial}{\partial r} \right. \quad (13.9)$$

308 We get:

$$\left(\frac{\partial V}{\partial r} \right)_{(z_0,r)} \approx \left(\frac{\partial V}{\partial r} \right)_{(z_0,0)} + \left(\frac{\partial^2 V}{\partial r^2} \right)_{(z_0,0)} \cdot r + \left(\frac{\partial V}{\partial r} \right)_{(z_0,0)} \cdot 1 \quad (13.10)$$

309 On axis Oz we can apply relation 13.3. That's why we can remove these
 310 two terms (first and third) from equation 13.10:

311 So we get (if $r = 0$):

$$\left(\frac{\partial V}{\partial r} \right)_{(z_0,r)} \approx \left(\frac{\partial^2 V}{\partial r^2} \right)_{(z_0,0)} \cdot r \quad (13.11)$$

312 We can now divide both sides by r .

$$\left| \cdot \frac{1}{r} \right. \quad (13.12)$$

313 We have relation, which has been published in Pierre Grivet's book[1].

$$\left(\frac{1}{r} \frac{\partial V}{\partial r} \right)_{(z_0,r)} \approx \left(\frac{\partial^2 V}{\partial r^2} \right)_{(z_0,0)} \quad (13.13)$$

314 Approximation of this term on numerical mesh has been already determined
 315 in previous subsection (13.6).

316 **13.8 Value of second partial derivative of V with respect to z on**
317 **axis O_z**

318 The third term of Laplace operator in rotationally symmetrical systems 13.2
319 takes form (on picture 10):

$$\left(\frac{\partial^2 V}{\partial z^2} \right)_{P_2} \approx \frac{\frac{V_3 - V_2}{h_z} - \frac{V_2 - V_1}{h_z}}{h_z} = \frac{V_1 + V_3 - 2V_2}{h_z^2} \quad (13.14)$$

320 Now we have determined all the 3 approximations o partial derivatives of V
321 in cylindrically symmetrical systems (on axis O_z).

322 **14 Partial derivatives off O_z axis**

323 **14.1 Personal note**

324 This is my personal interpretation. I cannot guarantee correctness of this ap-
325 proach

326 **14.2 Nodes numbering in Liebmann mesh (off axis O_z)**

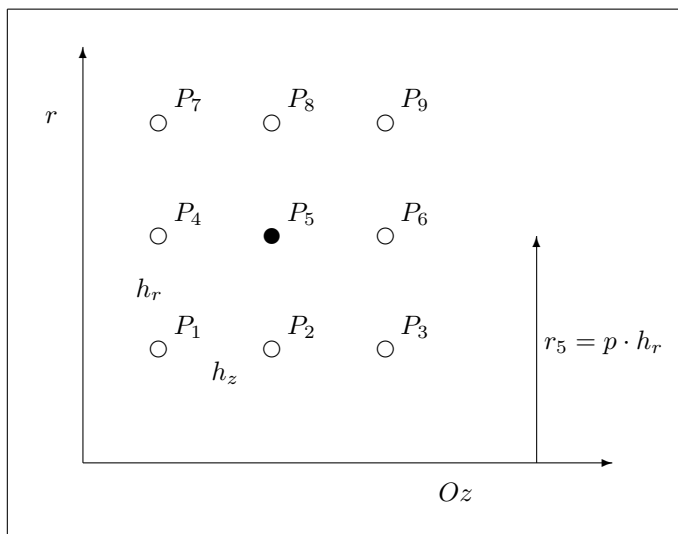


Figure 11: Nodes off axis O_z . Exemplary vector r_5 describes distance from axis O_z to node P_5

327 Mesh step in z direction is h_z . Mesh step in r direction is h_r . Sample mesh
328 points P_5 lies off O_z axis. Distance between mesh point P_5 and O_z axis is r_5 .

329 For ANSI C meshes (in Liebmann source code) the following relations have
330 place:

$$r = p h_r \quad (14.1)$$

$$p = i_{row} - 1 \quad (14.2)$$

331 **14.3 Taylor expansion of $V_{(z,r)}$ function in cylindrical coordinates**

332 Taylor expansion of $V_{(z,r)}$ function in cylindrical coordinates [3]:

$$\begin{aligned}
V_{(z,r)} = V_{(z_0,r_0)} + \left(\frac{\partial V}{\partial z} \right)_{(z_0,r_0)} (z - z_0) + \\
\left(\frac{\partial V}{\partial r} \right)_{(z_0,r_0)} (r - r_0) + \\
\frac{1}{2!} \left(\frac{\partial^2 V}{\partial z^2} \right)_{(z_0,r_0)} (z - z_0)^2 + \dots
\end{aligned} \tag{14.3}$$

333 **14.4 Laplace operator in rotationally symmetrical systems**

334 Laplace operator in cylindrical coordinates (cylindrical symmetry) consist of 3
335 elements [1] (on page 42):

$$\begin{aligned}
\nabla^2 (V_{(z,r)}) = \left(\frac{\partial^2 V}{\partial r^2} \right) + \\
\frac{1}{r} \left(\frac{\partial V}{\partial r} \right) + \\
\left(\frac{\partial^2 V}{\partial z^2} \right)
\end{aligned} \tag{14.4}$$

336 In this chapter we will try to determine approximation of each term.

337 **14.5 Value of second partial derivative of V with respect to r off**
338 **axis Oz**

$$\left(\frac{\partial^2 V}{\partial r^2} \right)_{P_5} \approx \frac{\frac{V_8 - V_5}{h_r} - \frac{V_5 - V_2}{h_r}}{h_r} = \frac{V_2 + V_8 - 2V_5}{h_r^2} \tag{14.5}$$

339 **14.6 Value of first partial derivative of V with respect to r divided**
340 **by r off axis Oz**

$$\frac{1}{r_5} \left(\frac{\partial V}{\partial r} \right)_{P_5} \approx \frac{1}{r_5} \frac{V_8 - V_2}{2h_r} = \frac{V_8 - V_2}{2r_5 h_z} \tag{14.6}$$

341 **14.7 Value of second partial derivative of V with respect to z off**
342 **axis Oz**

$$\left(\frac{\partial^2 V}{\partial z^2} \right)_{P_5} \approx \frac{\frac{V_6 - V_5}{h_z} - \frac{V_5 - V_4}{h_z}}{h_z} = \frac{V_4 + V_6 - 2V_5}{h_z^2} \tag{14.7}$$

343 **15 Relaxation formula for node P1 (on axis Oz)**

344 **15.1 Node description**

345 Left, bottom corner of mesh ZR (on axis Oz).

346 **15.2 Calculation of relaxation formula**

347 Laplace equation at node P_1

$$\nabla^2 (V_{(z,r)})_{P_1} = 0 \quad (15.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_1} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_1} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_1} = 0 \quad (15.2)$$

348 Approximation of partial derivatives of $V_{(z,r)}$ at node P_1

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_1} \approx \frac{\frac{V_4 - V_1}{h_r} - \frac{V_1 - V_4}{h_r}}{h_r} = \frac{2(V_4 - V_1)}{h_r^2} \quad (15.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_1} \approx \frac{2(V_4 - V_1)}{h_r^2} \quad (15.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_1} \approx \frac{\frac{V_2 - V_1}{h_z} - \frac{V_1 - V_{1z-}}{h_z}}{h_z} = \frac{V_2 - V_1}{h_z^2} - \frac{g_{1z-}}{h_z} \quad (15.5)$$

349 Let us substitute approximations to Laplace equation.

$$\frac{2(V_4 - V_1)}{h_r^2} + \frac{2(V_4 - V_1)}{h_r^2} + \frac{V_2 - V_1}{h_z^2} - \frac{g_{1z-}}{h_z} = 0 \quad (15.6)$$

$$\frac{2(V_4 - V_1)}{h_r^2} + \frac{2(V_4 - V_1)}{h_r^2} + \frac{V_2 - V_1}{h_z^2} = \frac{g_{1z-}}{h_z} \quad (15.7)$$

350 Let us find V_1

$$V_1 = ? \quad (15.8)$$

351 Let us multiply both sides

$$| \cdot h_z^2 h_r^2 \quad (15.9)$$

352 We obtain

$$2V_4 h_z^2 - 2V_1 h_z^2 + 2V_4 h_z^2 - 2V_1 h_z^2 + V_2 h_r^2 - V_1 h_r^2 = g_{1z-} h_z h_r^2 \quad (15.10)$$

353 Let us simplify this equation:

$$V_1 (2h_z^2 + 2h_z^2 + h_r^2) = 2V_4 h_z^2 + 2V_4 h_z^2 + V_2 h_r^2 - g_{1z-} h_z h_r^2 \quad (15.11)$$

354 So we have:

$$V_1 (4h_z^2 + h_r^2) = 4V_4 h_z^2 + V_2 h_r^2 - g_{1z-} h_z h_r^2 \quad (15.12)$$

355 15.3 Final forms of relaxation formula

356 15.3.1 zrLV_RELAX5_P1_ON_A

$$\begin{aligned} h_z &\neq h_r \\ g_{1z-} &\neq 0 \\ V_1 &= \frac{4V_4 h_z^2 + V_2 h_r^2 - g_{1z-} h_z h_r^2}{4h_z^2 + h_r^2} \end{aligned} \quad (15.13)$$

357 15.3.2 zrLV_RELAX5_P1_ON_B

$$\begin{aligned} h_z &\neq h_r \\ g_{1z-} &= 0 \\ V_1 &= \frac{4V_4 h_z^2 + V_2 h_r^2}{4h_z^2 + h_r^2} \end{aligned} \quad (15.14)$$

358 15.3.3 zrLV_RELAX5_P1_ON_C

$$\begin{aligned} h_z &= h_r = h \\ g_{1z-} &\neq 0 \\ V_1 &= \frac{4V_4 + V_2 - g_{1z-} h}{5} \end{aligned} \quad (15.15)$$

359 15.3.4 zrLV_RELAX5_P1_ON_D

$$\begin{aligned} h_z &= h_r = h \\ g_{1z-} &= 0 \\ V_1 &= \frac{4V_4 + V_2}{5} \end{aligned} \quad (15.16)$$

360 **16 Relaxation formula for node P2 (on axis Oz)**

361 **16.1 Node description**

362 Bottom edge of mesh ZR (on axis Oz).

363 **16.2 Calculation of relaxation formula**

364 Laplace equation at node P_2

$$\nabla^2 (V_{(z,r)})_{P_2} = 0 \quad (16.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_2} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_2} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_2} = 0 \quad (16.2)$$

365 Approximation of partial derivatives of $V_{(z,r)}$ at node P_2

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_2} \approx \frac{\frac{V_5 - V_2}{h_r} - \frac{V_2 - V_5}{h_r}}{h_r} = \frac{2(V_5 - V_2)}{h_r^2} \quad (16.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_2} \approx \frac{2(V_5 - V_2)}{h_r^2} \quad (16.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_2} \approx \frac{\frac{V_3 - V_2}{h_z} - \frac{V_2 - V_1}{h_z}}{h_z} = \frac{V_1 + V_3 - 2V_2}{h_z^2} \quad (16.5)$$

366 Let us substitute approximations to Laplace equation.

$$\frac{2(V_5 - V_2)}{h_r^2} + \frac{2(V_5 - V_2)}{h_r^2} + \frac{V_1 + V_3 - 2V_2}{h_z^2} = 0 \quad (16.6)$$

367 There are no g expressions to move, to formula 7 has identical form as
368 formula 6.

$$\frac{2(V_5 - V_2)}{h_r^2} + \frac{2(V_5 - V_2)}{h_r^2} + \frac{V_1 + V_3 - 2V_2}{h_z^2} = 0 \quad (16.7)$$

369 Let us find V_2

$$V_2 = ? \quad (16.8)$$

370 Let us multiply both sides

$$| \cdot h_z^2 h_r^2 \quad (16.9)$$

371 We obtain

$$2V_5 h_z^2 - 2V_2 h_z^2 + 2V_5 h_r^2 - 2V_2 h_r^2 + V_1 h_r^2 + V_3 h_r^2 - 2V_2 h_r^2 = 0 \quad (16.10)$$

372 Let us simplify this equation:

$$V_2 (2h_z^2 + 2h_z^2 + 2h_r^2) = 2V_5 h_z^2 + 2V_5 h_z^2 + V_1 h_r^2 + V_3 h_r^2 \quad (16.11)$$

373 So we have:

$$V_2 (4h_z^2 + 2h_r^2) = 4V_5 h_z^2 + (V_1 + V_3) h_r^2 \quad (16.12)$$

374 16.3 Final forms of relaxation formula

375 16.3.1 zrLV_RELAX5_P2_ON_A

$$h_z \neq h_r$$

$$V_2 = \frac{4V_5 h_z^2 + (V_1 + V_3) h_r^2}{4h_z^2 + 2h_r^2} \quad (16.13)$$

376 16.3.2 zrLV_RELAX5_P2_ON_B

$$h_z \neq h_r$$

$$V_2 = \frac{4V_5 h_z^2 + (V_1 + V_3) h_r^2}{4h_z^2 + 2h_r^2} \quad (16.14)$$

377 16.3.3 zrLV_RELAX5_P2_ON_C

$$h_z = h_r = h$$

$$V_2 = \frac{4V_5 + V_1 + V_3}{6} \quad (16.15)$$

378 16.3.4 zrLV_RELAX5_P2_ON_D

$$h_z = h_r = h$$

$$V_2 = \frac{4V_5 + V_1 + V_3}{6} \quad (16.16)$$

379 **17 Relaxation formula for node P3 (on axis Oz)**

380 **17.1 Node description**

381 Right, bottom corner of mesh ZR (on axis Oz).

382 **17.2 Calculation of relaxation formula**

383 Laplace equation at node P_3

$$\nabla^2 (V_{(z,r)})_{P_3} = 0 \quad (17.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_3} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_3} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_3} = 0 \quad (17.2)$$

384 Approximation of partial derivatives of $V_{(z,r)}$ at node P_3

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_3} \approx \frac{\frac{V_6 - V_3}{h_r} - \frac{V_3 - V_6}{h_r}}{h_r} = \frac{2(V_6 - V_3)}{h_r^2} \quad (17.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_3} \approx \frac{2(V_6 - V_3)}{h_r^2} \quad (17.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_3} \approx \frac{\frac{V_{3z+} - V_3}{h_z} - \frac{V_3 - V_2}{h_z}}{h_z} = \frac{V_2 - V_3}{h_z^2} + \frac{g_{3z+}}{h_z} \quad (17.5)$$

385 Let us substitute approximations to Laplace equation.

$$\frac{2(V_6 - V_3)}{h_r^2} + \frac{2(V_6 - V_3)}{h_r^2} + \frac{V_2 - V_3}{h_z^2} + \frac{g_{3z+}}{h_z} = 0 \quad (17.6)$$

$$\frac{2(V_6 - V_3)}{h_r^2} + \frac{2(V_6 - V_3)}{h_r^2} + \frac{V_2 - V_3}{h_z^2} = -\frac{g_{3z+}}{h_z} \quad (17.7)$$

386 Let us find V_3

$$V_3 = ? \quad (17.8)$$

387 Let us multiply both sides

$$| \cdot h_z^2 h_r^2 \quad (17.9)$$

388 We obtain

$$2V_6 h_z^2 - 2V_3 h_z^2 + 2V_6 h_z^2 - 2V_3 h_z^2 + V_2 h_r^2 - V_3 h_r^2 = -g_{3z+} h_z h_r^2 \quad (17.10)$$

389 Let us simplify this equation:

$$V_3 (2h_z^2 + 2h_z^2 + h_r^2) = 2V_6h_z^2 + 2V_6h_z^2 + V_2h_r^2 + g_{3z+}h_zh_r^2 \quad (17.11)$$

390 So we have:

$$V_3 (4h_z^2 + h_r^2) = 4V_6h_z^2 + V_2h_r^2 + g_{1z-}h_zh_r^2 \quad (17.12)$$

391 17.3 Final forms of relaxation formula

392 17.3.1 zrLV_RELAX5_P3_ON_A

$$\begin{aligned} & h_z \neq h_r \\ & g_{3z+} \neq 0 \\ 393 \quad & V_3 = \frac{4V_6h_z^2 + V_2h_r^2 + g_{3z+}h_zh_r^2}{4h_z^2 + h_r^2} \end{aligned} \quad (17.13)$$

394 17.3.2 zrLV_RELAX5_P3_ON_B

$$\begin{aligned} & h_z \neq h_r \\ & g_{3z+} = 0 \\ & V_3 = \frac{4V_6h_z^2 + V_2h_r^2}{4h_z^2 + h_r^2} \end{aligned} \quad (17.14)$$

395 17.3.3 zrLV_RELAX5_P3_ON_C

$$\begin{aligned} & h_z = h_r = h \\ & g_{3z+} \neq 0 \\ & V_3 = \frac{4V_6 + V_2 + g_{3z+}h}{5} \end{aligned} \quad (17.15)$$

396 17.3.4 zrLV_RELAX5_P3_ON_D

$$\begin{aligned} & h_z = h_r = h \\ & g_{3z+} = 0 \\ & V_3 = \frac{4V_6 + V_2}{5} \end{aligned} \quad (17.16)$$

397 **18 Relaxation formula for node P4**

398 **18.1 Node description**

399 Left edge of mesh ZR.

400 **18.2 Calculation of relaxation formula**

401 Laplace equation at node P_4

$$\nabla^2 (V_{(z,r)})_{P_4} = 0 \quad (18.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_4} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_4} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_4} = 0 \quad (18.2)$$

402 Approximation of partial derivatives of $V_{(z,r)}$ at node P_4

403

404 (note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_4} \approx \frac{\frac{V_7 - V_4}{h_r} - \frac{V_4 - V_1}{h_r}}{h_r} = \frac{V_1 + V_7 - 2V_4}{h_r^2} \quad (18.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_4} \approx \frac{1}{r} \frac{V_7 - V_1}{2h_r} = \frac{V_7 - V_1}{2ph_r^2} \quad (18.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_4} \approx \frac{\frac{V_5 - V_1}{h_z} - \frac{V_4 - V_{4z-}}{h_z}}{h_z} = \frac{V_5 - V_4}{h_z^2} - \frac{g_{4z-}}{h_z} \quad (18.5)$$

405 Let us substitute approximations to Laplace equation.

$$\frac{V_1 + V_7 - 2V_4}{h_r^2} + \frac{V_7 - V_1}{2ph_r^2} + \frac{V_5 - V_4}{h_z^2} - \frac{g_{4z-}}{h_z} = 0 \quad (18.6)$$

$$\frac{V_1 + V_7 - 2V_4}{h_r^2} + \frac{V_7 - V_1}{2ph_r^2} + \frac{V_5 - V_4}{h_z^2} = \frac{g_{4z-}}{h_z} \quad (18.7)$$

406 Let us find V_4

$$V_4 = ? \quad (18.8)$$

407 Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \quad (18.9)$$

408 We obtain

$$2pV_1h_z^2 + 2pV_7h_z^2 - 4pV_4h_z^2 + V_7h_z^2 - V_1h_z^2 + 2pV_5h_r^2 - 2pV_4h_r^2 = 2pg_{4z-}h_zh_r^2 \quad (18.10)$$

409 Let us simplify this equation:

$$V_4(4ph_z^2 + 2ph_r^2) = V_1(2ph_z^2 - h_z^2) + V_7(2ph_z^2 + h_z^2) + V_52ph_r^2 - 2pg_{4z-}h_zh_r^2 \quad (18.11)$$

410 So we have:

$$V_42p(2h_z^2 + h_r^2) = V_1h_z^2(2p-1) + V_7h_z^2(2p+1) + V_52ph_r^2 - 2pg_{4z-}h_zh_r^2 \quad (18.12)$$

411 18.3 Final forms of relaxation formula

412 18.3.1 zrLV_RELAX5_P4_A

$$h_z \neq h_r$$

$$g_{4z-} \neq 0$$

$$413 \quad V_4 = \frac{h_z^2(2p-1)}{2p(2h_z^2 + h_r^2)}V_1 + \frac{h_z^2(2p+1)}{2p(2h_z^2 + h_r^2)}V_7 + \frac{h_r^2}{2h_z^2 + h_r^2}V_5 - \frac{g_{4z-}h_zh_r^2}{2h_z^2 + h_r^2} \quad (18.13)$$

414 18.3.2 zrLV_RELAX5_P4_B

$$h_z \neq h_r$$

$$g_{4z-} = 0$$

$$415 \quad V_4 = \frac{h_z^2(2p-1)}{2p(2h_z^2 + h_r^2)}V_1 + \frac{h_z^2(2p+1)}{2p(2h_z^2 + h_r^2)}V_7 + \frac{h_r^2}{2h_z^2 + h_r^2}V_5 \quad (18.14)$$

416 18.3.3 zrLV_RELAX5_P4_C

$$h_z = h_r = h$$

$$g_{4z-} \neq 0$$

$$417 \quad V_4 = \frac{2p-1}{6p}V_1 + \frac{2p+1}{6p}V_7 + \frac{1}{3}V_5 - \frac{g_{4z-}h}{3} \quad (18.15)$$

418 18.3.4 zrLV_RELAX5_P4_D

$$h_z = h_r = h$$

$$g_{4z-} = 0$$

$$V_4 = \frac{2p-1}{6p}V_1 + \frac{2p+1}{6p}V_7 + \frac{1}{3}V_5 \quad (18.16)$$

419 **19 Relaxation formula for node P5**

420 **19.1 Node description**

421 Inner node of mesh ZR.

422 **19.2 Calculation of relaxation formula**

423 Laplace equation at node P_5

$$\nabla^2 (V_{(z,r)})_{P_5} = 0 \quad (19.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_5} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_5} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_5} = 0 \quad (19.2)$$

424 Approximation of partial derivatives of $V_{(z,r)}$ at node P_5

425

426 (note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_5} \approx \frac{\frac{V_8 - V_5}{h_r} - \frac{V_5 - V_2}{h_r}}{h_r} = \frac{V_2 + V_8 - 2V_5}{h_r^2} \quad (19.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_5} \approx \frac{1}{r} \frac{V_8 - V_2}{2h_r} = \frac{V_8 - V_2}{2ph_r^2} \quad (19.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_5} \approx \frac{\frac{V_6 - V_5}{h_z} - \frac{V_5 - V_4}{h_z}}{h_z} = \frac{V_4 + V_6 - 2V_5}{h_z^2} \quad (19.5)$$

427 Let us substitute approximations to Laplace equation.

$$\frac{V_2 + V_8 - 2V_5}{h_r^2} + \frac{V_8 - V_2}{2ph_r^2} + \frac{V_4 + V_6 - 2V_5}{h_z^2} = 0 \quad (19.6)$$

428 We don't need to simplify this equation in step 7:

$$\frac{V_2 + V_8 - 2V_5}{h_r^2} + \frac{V_8 - V_2}{2ph_r^2} + \frac{V_4 + V_6 - 2V_5}{h_z^2} = 0 \quad (19.7)$$

429 Let us find V_5

$$V_5 = ? \quad (19.8)$$

430 Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \quad (19.9)$$

431 We obtain

$$2pV_2h_z^2 + 2pV_8h_z^2 - 4pV_5h_z^2 + V_8h_z^2 - V_2h_z^2 + 2pV_4h_r^2 + 2pV_6h_r^2 - 4pV_5h_r^2 = 0 \quad (19.10)$$

432 Let us simplify this equation:

$$V_5(4ph_z^2 + 4ph_r^2) = V_2(2ph_z^2 - h_z^2) + V_8(2ph_z^2 + h_z^2) + 2ph_r^2V_4 + 2ph_r^2V_6 \quad (19.11)$$

433 So we have:

$$V_54p(h_z^2 + h_r^2) = V_2h_z^2(2p - 1) + V_8h_z^2(2p + 1) + V_42ph_r^2 + V_62ph_r^2 \quad (19.12)$$

434 19.3 Final forms of relaxation formula

435 19.3.1 zrLV_RELAX5_P5_A

$$436 \quad h_z \neq h_r$$

$$V_5 = \frac{h_z^2(2p - 1)}{4p(h_z^2 + h_r^2)}V_2 + \frac{h_z^2(2p + 1)}{4p(h_z^2 + h_r^2)}V_8 + \frac{h_r^2}{2(h_z^2 + h_r^2)}(V_4 + V_6) \quad (19.13)$$

437 19.3.2 zrLV_RELAX5_P5_B

438 This formula is identical to formula A:

$$439 \quad h_z \neq h_r$$

$$V_5 = \frac{h_z^2(2p - 1)}{4p(h_z^2 + h_r^2)}V_2 + \frac{h_z^2(2p + 1)}{4p(h_z^2 + h_r^2)}V_8 + \frac{h_r^2}{2(h_z^2 + h_r^2)}(V_4 + V_6) \quad (19.14)$$

440 19.3.3 zrLV_RELAX5_P5_C

$$441 \quad h_z = h_r = h$$

$$V_5 = \frac{2p - 1}{8p}V_2 + \frac{2p + 1}{8p}V_8 + \frac{1}{4}(V_4 + V_6) \quad (19.15)$$

442 19.3.4 zrLV_RELAX5_P5_D

443 This formula is identical to formula C:

$$444 \quad h_z = h_r = h$$

$$V_5 = \frac{2p - 1}{8p}V_2 + \frac{2p + 1}{8p}V_8 + \frac{1}{4}(V_4 + V_6) \quad (19.16)$$

445 **20 Relaxation formula for node P6**

446 **20.1 Node description**

447 Right edge of mesh ZR.

448 **20.2 Calculation of relaxation formula**

449 Laplace equation at node P_6

$$\nabla^2 (V_{(z,r)})_{P_6} = 0 \quad (20.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_6} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_6} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_6} = 0 \quad (20.2)$$

450 Approximation of partial derivatives of $V_{(z,r)}$ at node P_6

451

452 (note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_6} \approx \frac{\frac{V_9 - V_6}{h_r} - \frac{V_6 - V_3}{h_r}}{h_r} = \frac{V_3 + V_9 - 2V_6}{h_r^2} \quad (20.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_6} \approx \frac{1}{r} \frac{V_9 - V_3}{2h_r} = \frac{V_9 - V_3}{2ph_r^2} \quad (20.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_6} \approx \frac{\frac{V_{6z+} - V_6}{h_z} - \frac{V_6 - V_5}{h_z}}{h_z} = \frac{V_5 - V_6}{h_z^2} + \frac{g_{6z+}}{h_z} \quad (20.5)$$

453 Let us substitute approximations to Laplace equation.

$$\frac{V_3 + V_9 - 2V_6}{h_r^2} + \frac{V_9 - V_3}{2ph_r^2} + \frac{V_5 - V_6}{h_z^2} + \frac{g_{6z+}}{h_z} = 0 \quad (20.6)$$

$$\frac{V_3 + V_9 - 2V_6}{h_r^2} + \frac{V_9 - V_3}{2ph_r^2} + \frac{V_5 - V_6}{h_z^2} = -\frac{g_{6z+}}{h_z} \quad (20.7)$$

454 Let us find V_6

$$V_6 = ? \quad (20.8)$$

455 Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \quad (20.9)$$

456 We obtain

$$2pV_3h_z^2 + 2pV_9h_z^2 - 4pV_6h_z^2 + V_9h_z^2 - V_3h_z^2 + 2pV_5h_r^2 - 2pV_6h_r^2 = -2pg_{6z+}h_zh_r^2 \quad (20.10)$$

457 Let us simplify this equation:

$$V_6(4ph_z^2 + 2ph_r^2) = V_3(2ph_z^2 - h_z^2) + V_9(2ph_z^2 + h_z^2) + V_52ph_r^2 + 2pg_{6z+}h_zh_r^2 \quad (20.11)$$

458 So we have:

$$V_62p(2h_z^2 + h_r^2) = V_3h_z^2(2p-1) + V_9h_z^2(2p+1) + V_52ph_r^2 + 2pg_{6z+}h_zh_r^2 \quad (20.12)$$

459 20.3 Final forms of relaxation formula

460 20.3.1 zrLV_RELAX5_P6_A

$$h_z \neq h_r$$

$$g_{6z+} \neq 0$$

$$461 V_6 = \frac{h_z^2(2p-1)}{2p(2h_z^2 + h_r^2)}V_3 + \frac{h_z^2(2p+1)}{2p(2h_z^2 + h_r^2)}V_9 + \frac{h_r^2}{2h_z^2 + h_r^2}V_5 + \frac{g_{6z+}h_zh_r^2}{2h_z^2 + h_r^2} \quad (20.13)$$

462 20.3.2 zrLV_RELAX5_P6_B

$$h_z \neq h_r$$

$$g_{6z+-} = 0$$

$$463 V_6 = \frac{h_z^2(2p-1)}{2p(2h_z^2 + h_r^2)}V_3 + \frac{h_z^2(2p+1)}{2p(2h_z^2 + h_r^2)}V_9 + \frac{h_r^2}{2h_z^2 + h_r^2}V_5 \quad (20.14)$$

464 20.3.3 zrLV_RELAX5_P6_C

$$h_z = h_r = h$$

$$g_{6z+} \neq 0$$

$$465 V_6 = \frac{2p-1}{6p}V_3 + \frac{2p+1}{6p}V_9 + \frac{1}{3}V_5 + \frac{g_{6z+}h}{3} \quad (20.15)$$

466 20.3.4 zrLV_RELAX5_P6_D

$$h_z = h_r = h$$

$$g_{6z+} = 0$$

$$467 V_4 = \frac{2p-1}{6p}V_3 + \frac{2p+1}{6p}V_9 + \frac{1}{3}V_5 \quad (20.16)$$

468 **21 Relaxation formula for node P7**

469 **21.1 Node description**

470 Left, upper corner of mesh ZR.

471 **21.2 Calculation of relaxation formula**

472 Laplace equation at node P_7

$$\nabla^2 (V_{(z,r)})_{P_7} = 0 \quad (21.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_7} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_7} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_7} = 0 \quad (21.2)$$

473 Approximation of partial derivatives of $V_{(z,r)}$ at node P_7

474

475 (note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_7} \approx \frac{\frac{V_{7r+} - V_7}{h_r} - \frac{V_7 - V_4}{h_r}}{h_r} = \frac{V_4 - V_7}{h_r^2} + \frac{g_{7r+}}{h_r} \quad (21.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_7} \approx \frac{1}{r} \frac{V_{7r+} - V_4}{2h_r} = \frac{V_7 + g_{7r+}h_r - V_4}{2ph_r^2} = \frac{V_7 - V_4}{2ph_r^2} + \frac{g_{7r+}}{2ph_r} \quad (21.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_7} \approx \frac{\frac{V_8 - V_7}{h_z} - \frac{V_7 - V_{7z-}}{h_z}}{h_z} = \frac{V_8 - V_7}{h_z^2} - \frac{g_{7z-}}{h_z} \quad (21.5)$$

476 Let us substitute approximations to Laplace equation.

$$\frac{V_4 - V_7}{h_r^2} + \frac{g_{7r+}}{h_r} + \frac{V_7 - V_4}{2ph_r^2} + \frac{g_{7r+}}{2ph_r} + \frac{V_8 - V_7}{h_z^2} - \frac{g_{7z-}}{h_z} = 0 \quad (21.6)$$

$$\frac{V_4 - V_7}{h_r^2} + \frac{V_7 - V_4}{2ph_r^2} + \frac{V_8 - V_7}{h_z^2} = -\frac{g_{7r+}}{h_r} - \frac{g_{7r+}}{2ph_r} + \frac{g_{7z-}}{h_z} \quad (21.7)$$

477 Let us find V_7

$$V_7 = ? \quad (21.8)$$

478 Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \quad (21.9)$$

479 We obtain

$$2pV_4 h_z^2 - 2pV_7 h_z^2 + V_7 h_z^2 - V_4 h_z^2 + 2pV_8 h_r^2 - 2pV_7 h_r^2 = \quad (21.10)$$

$$-2pg_{7r+} h_z^2 h_r - g_{7r+} h_z^2 h_r + 2pg_{7z-} h_z h_r^2$$

480 Let us simplify this equation:

$$V_7 (2ph_z^2 - h_z^2 + 2ph_r^2) = V_4 (2ph_z^2 - h_z^2) + V_8 (2ph_r^2) + \quad (21.11)$$

$$2pg_{7r+} h_z^2 h_r + g_{7r+} h_z^2 h_r - 2pg_{7z-} h_z h_r^2$$

481 So we have:

$$V_7 ((2p-1)h_z^2 + 2ph_r^2) = V_4 h_z^2 (2p-1) + V_8 2ph_r^2 + \quad (21.12)$$

$$2pg_{7r+} h_z^2 h_r + g_{7r+} h_z^2 h_r - 2pg_{7z-} h_z h_r^2$$

482 21.3 Final forms of relaxation formula

483 21.3.1 zrLV_RELAX5_P7_A

$$h_z \neq h_r$$

$$g_{7z-} \neq 0$$

$$g_{7r+} \neq 0$$

484

$$V_7 = \frac{h_z^2 (2p-1)}{(2p-1)h_z^2 + 2ph_r^2} V_4 + \frac{2ph_r^2}{(2p-1)h_z^2 + 2ph_r^2} V_8 + \quad (21.13)$$

$$\frac{(2p+1)g_{7r+} h_z^2 h_r - 2pg_{7z-} h_z h_r^2}{(2p-1)h_z^2 + 2ph_r^2}$$

485 21.3.2 zrLV_RELAX5_P7_B

$$h_z \neq h_r$$

$$g_{7z-} = 0$$

$$g_{7r+} = 0$$

$$V_7 = \frac{h_z^2 (2p-1)}{(2p-1)h_z^2 + 2ph_r^2} V_4 + \frac{2ph_r^2}{(2p-1)h_z^2 + 2ph_r^2} V_8 \quad (21.14)$$

486 **21.3.3 zrLV_RELAX5_P7_C**

$$h_z = h_r = h$$

$$g_{7z-} \neq 0$$

$$g_{7r+} \neq 0$$

487

$$V_7 = \frac{2p-1}{4p-1}V_4 + \frac{2p}{4p-1}V_8 + \frac{h((2p+1)g_{7r+} - g_{7z-})}{4p-1} \quad (21.15)$$

488 **21.3.4 zrLV_RELAX5_P7_D**

$$h_z = h_r = h$$

$$g_{7z-} = 0$$

$$g_{7r+} = 0$$

489

$$V_7 = \frac{2p-1}{4p-1}V_4 + \frac{2p}{4p-1}V_8 \quad (21.16)$$

490 **22 Relaxation formula for node P8**

491 **22.1 Node description**

492 Upper edge of mesh ZR.

493 **22.2 Calculation of relaxation formula**

494 Laplace equation at node P_8

$$\nabla^2 (V_{(z,r)})_{P_8} = 0 \quad (22.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_8} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_8} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_8} = 0 \quad (22.2)$$

495 Approximation of partial derivatives of $V_{(z,r)}$ at node P_8

496

497 (note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_8} \approx \frac{\frac{V_{8r+} - V_8}{h_r} - \frac{V_8 - V_5}{h_r}}{h_r} = \frac{V_5 - V_8}{h_r^2} + \frac{g_{8r+}}{h_r} \quad (22.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_8} \approx \frac{1}{r} \frac{V_{8r+} - V_5}{2h_r} = \frac{V_8 + g_{8r+}h_r - V_5}{2ph_r^2} = \frac{V_8 - V_5}{2ph_r^2} + \frac{g_{8r+}}{2ph_r} \quad (22.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_8} \approx \frac{\frac{V_9 - V_8}{h_z} - \frac{V_8 - V_7}{h_z}}{h_z} = \frac{V_7 + V_9 - 2V_8}{h_z^2} \quad (22.5)$$

498 Let us substitute approximations to Laplace equation.

$$\frac{V_5 - V_8}{h_r^2} + \frac{g_{8r+}}{h_r} + \frac{V_8 - V_5}{2ph_r^2} + \frac{g_{8r+}}{2ph_r} + \frac{V_7 + V_9 - 2V_8}{h_z^2} = 0 \quad (22.6)$$

$$\frac{V_5 - V_8}{h_r^2} + \frac{V_8 - V_5}{2ph_r^2} + \frac{V_7 + V_9 - 2V_8}{h_z^2} = -\frac{g_{8r+}}{h_r} - \frac{g_{8r+}}{2ph_r} \quad (22.7)$$

499 Let us find V_8

$$V_8 = ? \quad (22.8)$$

500 Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \quad (22.9)$$

501 We obtain

$$2pV_5 h_z^2 - 2pV_8 h_z^2 + V_8 h_z^2 - V_5 h_z^2 + 2pV_7 h_r^2 + 2pV_9 h_r^2 - 4pV_8 h_r^2 = -2pg_{8r+} h_z^2 h_r - g_{8r+} h_z^2 h_r \quad (22.10)$$

502 Let us simplify this equation:

$$V_8 (2ph_z^2 - h_z^2 + 4ph_r^2) = V_5 (2ph_z^2 - h_z^2) + (V_7 + V_9) 2ph_r^2 + 2pg_{8r+} h_z^2 h_r + g_{8r+} h_z^2 h_r \quad (22.11)$$

503 So we have:

$$V_8 ((2p-1)h_z^2 + 4ph_r^2) = V_5 h_z^2 (2p-1) + (V_7 + V_9) 2ph_r^2 + 2pg_{8r+} h_z^2 h_r + g_{8r+} h_z^2 h_r \quad (22.12)$$

504 22.3 Final forms of relaxation formula

505 22.3.1 zrLV_RELAX5_P8_A

$$h_z \neq h_r$$

$$g_{8r+} \neq 0$$

$$V_8 = \frac{h_z^2 (2p-1)}{(2p-1)h_z^2 + 4ph_r^2} V_5 + \frac{2ph_r^2}{(2p-1)h_z^2 + 4ph_r^2} (V_7 + V_9) + \frac{(2p+1)h_z^2 h_r g_{8r+}}{(2p-1)h_z^2 + 4ph_r^2} \quad (22.13)$$

507 22.3.2 zrLV_RELAX5_P8_B

$$h_z \neq h_r$$

$$g_{8r+} = 0$$

$$V_8 = \frac{h_z^2 (2p-1)}{(2p-1)h_z^2 + 4ph_r^2} V_5 + \frac{2ph_r^2}{(2p-1)h_z^2 + 4ph_r^2} (V_7 + V_9) \quad (22.14)$$

509 22.3.3 zrLV_RELAX5_P8_C

$$h_z = h_r = h$$

$$g_{8r+} \neq 0$$

$$V_8 = \frac{2p-1}{6p-1} V_5 + \frac{2p}{6p-1} (V_7 + V_9) + \frac{(2p+1)h g_{8r+}}{6p-1} \quad (22.15)$$

511 **22.3.4 zrLV_RELAX5_P8_D**

$$h_z = h_r = h$$

$$g_{8r+} = 0$$

512

$$V_8 = \frac{2p-1}{6p-1}V_5 + \frac{2p}{6p-1}(V_7 + V_9) \quad (22.16)$$

513 **23 Relaxation formula for node P9**

514 **23.1 Node description**

515 Right, upper corner of mesh ZR.

516 **23.2 Calculation of relaxation formula**

517 Laplace equation at node P_9

$$\nabla^2 (V_{(z,r)})_{P_9} = 0 \quad (23.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_9} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_9} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_9} = 0 \quad (23.2)$$

518 Approximation of partial derivatives of $V_{(z,r)}$ at node P_9

519

520 (note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_9} \approx \frac{V_{9r+} - V_9}{h_r} - \frac{V_9 - V_6}{h_r} = \frac{V_6 - V_9}{h_r^2} + \frac{g_{9r+}}{h_r} \quad (23.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_9} \approx \frac{1}{r} \frac{V_{9r+} - V_6}{2h_r} = \frac{V_9 + g_{9r+}h_r - V_6}{2ph_r^2} = \frac{V_9 - V_6}{2ph_r^2} + \frac{g_{9r+}}{2ph_r} \quad (23.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_9} \approx \frac{V_{9z+} - V_9}{h_z} - \frac{V_9 - V_8}{h_z} = \frac{g_{9z+}}{h_z} + \frac{V_8 - V_9}{h_z^2} \quad (23.5)$$

521 Let us substitute approximations to Laplace equation.

$$\frac{V_6 - V_9}{h_r^2} + \frac{g_{9r+}}{h_r} + \frac{V_9 - V_6}{2ph_r^2} + \frac{g_{9r+}}{2ph_r} + \frac{V_8 - V_9}{h_z^2} + \frac{g_{9z+}}{h_z} = 0 \quad (23.6)$$

$$\frac{V_6 - V_9}{h_r^2} + \frac{V_9 - V_6}{2ph_r^2} + \frac{V_8 - V_9}{h_z^2} = -\frac{g_{9r+}}{h_r} - \frac{g_{9r+}}{2ph_r} - \frac{g_{9z+}}{h_z} \quad (23.7)$$

522 Let us find V_9

$$V_9 = ? \quad (23.8)$$

523 Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \quad (23.9)$$

524 We obtain

$$2pV_6 h_z^2 - 2pV_9 h_z^2 + V_9 h_z^2 - V_6 h_z^2 + 2pV_8 h_r^2 - 2pV_9 h_r^2 = \quad (23.10)$$

$$-2pg_{9r+} h_z^2 h_r - g_{9r+} h_z^2 h_r - 2pg_{9z+} h_z h_r^2$$

525 Let us simplify this equation:

$$V_9 (2ph_z^2 - h_z^2 + 2ph_r^2) = V_6 (2ph_z^2 - h_z^2) + V_8 (2ph_r^2) + \quad (23.11)$$

$$2pg_{9r+} h_z^2 h_r + g_{9r+} h_z^2 h_r + 2pg_{9z+} h_z h_r^2$$

526 So we have:

$$V_9 ((2p-1)h_z^2 + 2ph_r^2) = V_6 h_z^2 (2p-1) + V_8 2ph_r^2 + \quad (23.12)$$

$$(2p+1)g_{9r+} h_z^2 h_r + 2pg_{9z+} h_z h_r^2$$

527 23.3 Final forms of relaxation formula

528 23.3.1 zrLV_RELAX5_P9_A

$$h_z \neq h_r$$

$$g_{9z-} \neq 0$$

$$g_{9r+} \neq 0$$

529

$$V_9 = \frac{h_z^2 (2p-1)}{(2p-1)h_z^2 + 2ph_r^2} V_6 + \frac{2ph_r^2}{(2p-1)h_z^2 + 2ph_r^2} V_8 + \quad (23.13)$$

$$\frac{(2p+1)g_{9r+} h_z^2 h_r + 2pg_{9z+} h_z h_r^2}{(2p-1)h_z^2 + 2ph_r^2}$$

530 23.3.2 zrLV_RELAX5_P9_B

$$h_z \neq h_r$$

$$g_{9z-} = 0$$

$$g_{9r+} = 0$$

$$V_9 = \frac{h_z^2 (2p-1)}{(2p-1)h_z^2 + 2ph_r^2} V_6 + \frac{2ph_r^2}{(2p-1)h_z^2 + 2ph_r^2} V_8 \quad (23.14)$$

531 **23.3.3 zrLV_RELAX5_P9_C**

$$h_z = h_r = h$$

$$g_{9z-} \neq 0$$

$$g_{9r+} \neq 0$$

532

$$V_9 = \frac{2p-1}{4p-1}V_6 + \frac{2p}{4p-1}V_8 + \frac{h((2p+1)g_{9r+} + g_{9z+})}{4p-1} \quad (23.15)$$

533 **23.3.4 zrLV_RELAX5_P9_D**

$$h_z = h_r = h$$

$$g_{9z-} = 0$$

$$g_{9r+} = 0$$

534

$$V_9 = \frac{2p-1}{4p-1}V_6 + \frac{2p}{4p-1}V_8 \quad (23.16)$$

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