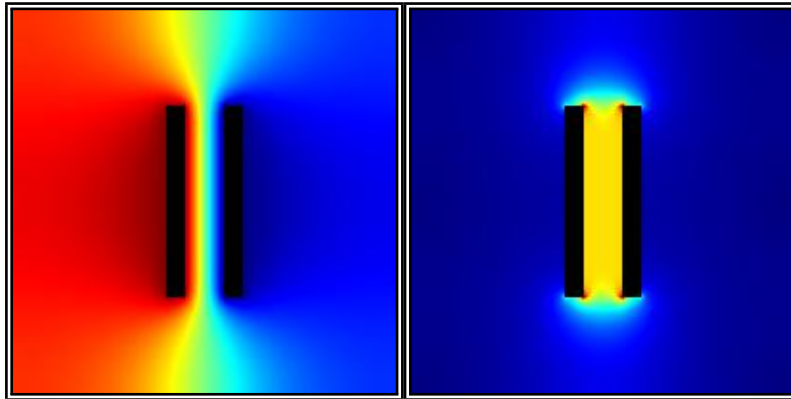


1

Liebmann technical documentation



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3

Laplace equation 2D (XY)
(Cartesian coordinates)
relaxation scheme explained
(5 - point star)

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project homepage: http://marcinkulbaka.prv.pl/Liebmann/index_en.html

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11

version 9

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2024.07.18

13

University of Maria Curie - Skłodowska in Lublin, Poland

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99 **1 Liebmann technical documentation series**

- 100 1. Wyznaczanie rozkładu pola elektrostatycznego w próżni metodą relak-
101 sacyjną Liebmann. (Polish version / wersja polska)
- 102 2. Determination of electrostatic field distribution by using Liebmann relax-
103 ation method. (English version / wersja angielska)
- 104 3. Graphics. Mapping voltages to colours (colormaps).
- 105 4. Laplace equation 2D (XY). (Cartesian coordinates). Relaxation scheme
106 explained. (5 - point star)
- 107 5. Laplace equation 2D (ZR) (Cylindrical coordinates). Relaxation scheme
108 explained. (5 - point star)
- 109 6. Liebmann source code. (ANSI C programming language)

110 **2 Versions of this document**

- 111 1. version 1 - 2023.11.03
- 112 2. version 2 - 2024.01.26
- 113 3. version 3 - 2024.02.02
- 114 4. version 4 - 2024.02.05
- 115 5. version 5 - 2024.05.18
- 116 6. version 6 - 2024.05.23
- 117 7. version 7 - 2024.05.24
- 118 8. version 8 - 2024.07.17
- 119 9. version 9 - 2024.07.18

120 **3 Solving Laplace equation using relaxation method**

121 I tried to solve Laplace equation using mainly information from Pierre Grivet's
122 book (Electron Optics) - [1].

123 There are few editions of this book (1965, 1972). Second edition (1972) con-
124 tains explanation of relaxation method (page 38).

125 More generalized approaches has been drafted by James R. Nagel - [2].
126 <https://my.ece.utah.edu/~ece6340/LECTURES/Feb1/> (visited 2023-03-01).

127

128 There are also publications edited by Albert Septier: Focusing of Charged
129 Particles [3] and Applied Charged Particle Optics (part A). [4].

130 I have also found some ideas in publication of D W O Heddle: Electrostatic
131 Lens Systems [5] (especially using PC computers to solve electrostatic prob-
132 lems).

133 I have also found (brief) description of by - hand solving of Laplace equa-
134 tion by Bohdan Paszkowski - [6] (Polish edition). English translation of this book
135 also exists - [7].

136
137 I would like to thank many people, who helped me with this challenge. Espe-
138 cially prof. dr hab. Mieczysław Jałochowski (supervisor of my master's thesis),
139 who enabled me to use SIMION and MATLAB software while writing master's
140 thesis about electron optical systems at University of Maria Curie - Skłodowska
141 in Lublin in 2008. I would also thank to prof. Marcin Turek for fruitful discus-
142 sion about numerical methods. What is more, my colleague Bartosz in 2012
143 had explained me general problems with software efficiency. So he had also
144 contributed significantly to the idea of Liebmann software (especially using C
145 language).

146 **4 Explanation of symbols in calculations**

- 147 • P_i - i -th mesh node
- 148 • V_i - value of electrostatic potential at node P_i . Unit - [V]
- 149 • h - mesh step (for example h_x - mesh step in x direction). Unit - [mm]
- 150 • $g_{i+/-}$ - gradient in direction i (for example $g_{1x-} = \frac{V_1 - V_{1x-}}{h_x}$. Unit - $[\frac{V}{mm}]$)
- 151 • i_{row} - index of row in mesh. Values of $i_{row} = 1, 2, \dots, \text{size_row}$
- 152 • i_{col} - index of column in mesh. Values of $i_{col} = 1, 2, \dots, \text{size_col}$

153 **5 Mesh XY - type A**

154 $h_x \neq h_y$

155 gradient V outside a mesh exists

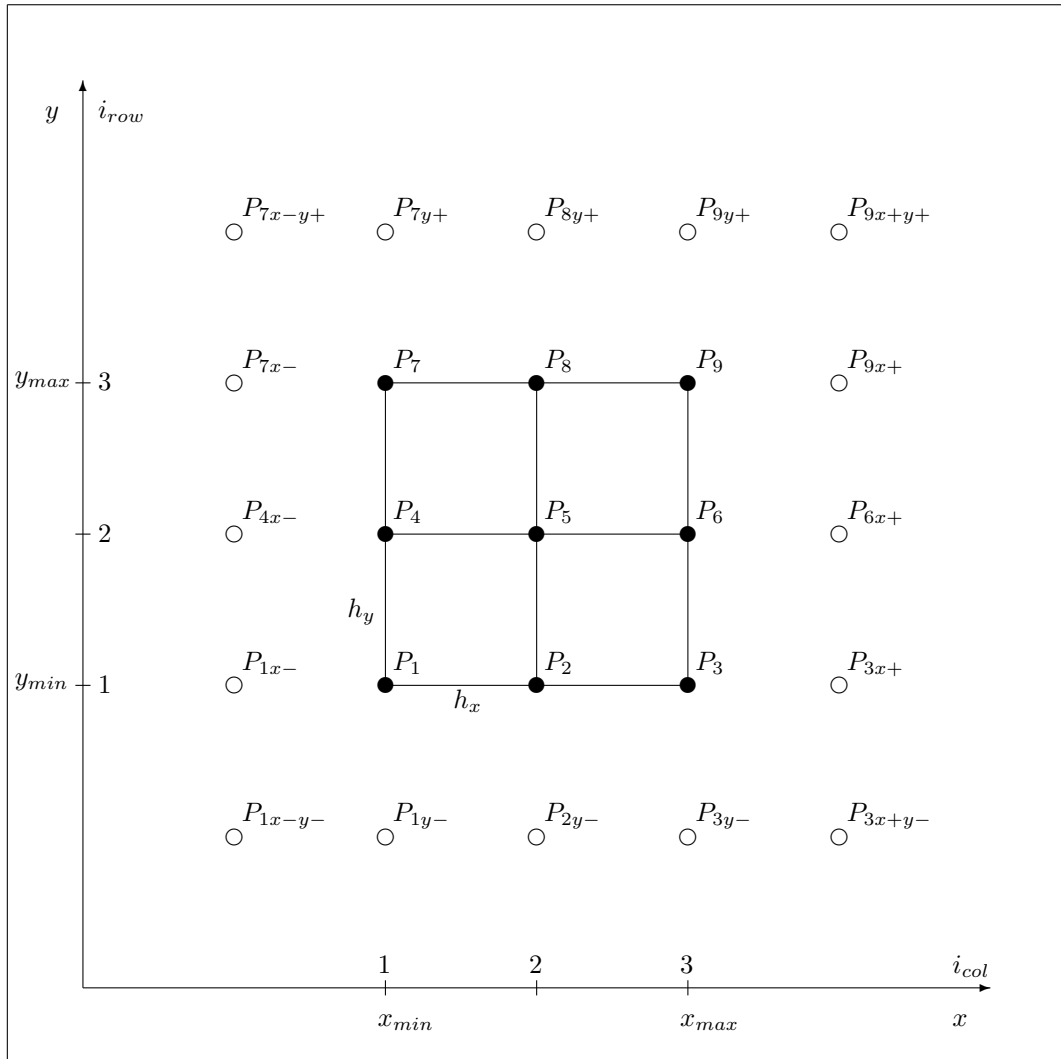


Figure 1: Mesh XY type A

156 **6 Mesh XY - type B**

157 $h_x \neq h_y$

158 gradient V outside a mesh does not exist

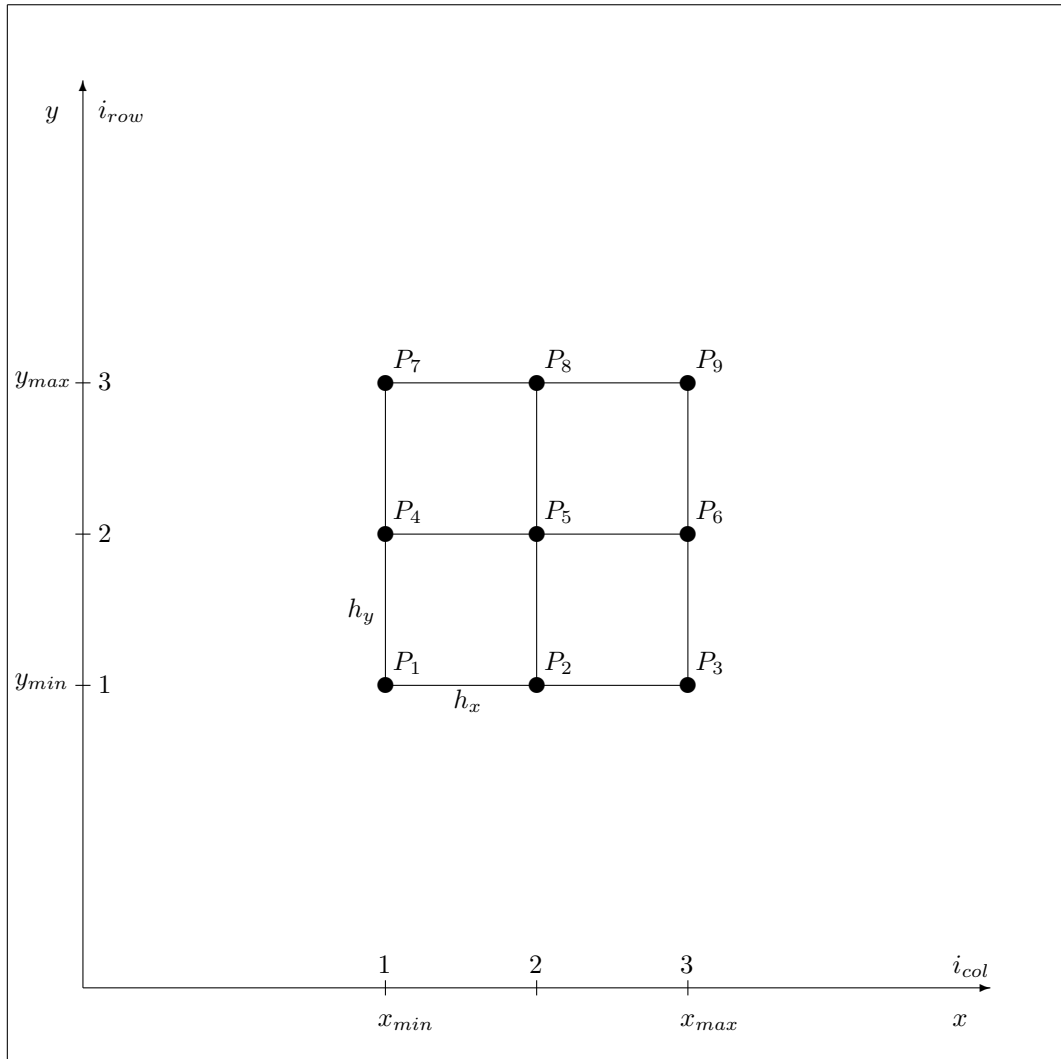


Figure 2: Mesh XY type B

159 **7 Mesh XY - type C**

160 $h_x = h_y = h$

161 gradient V outside a mesh exists

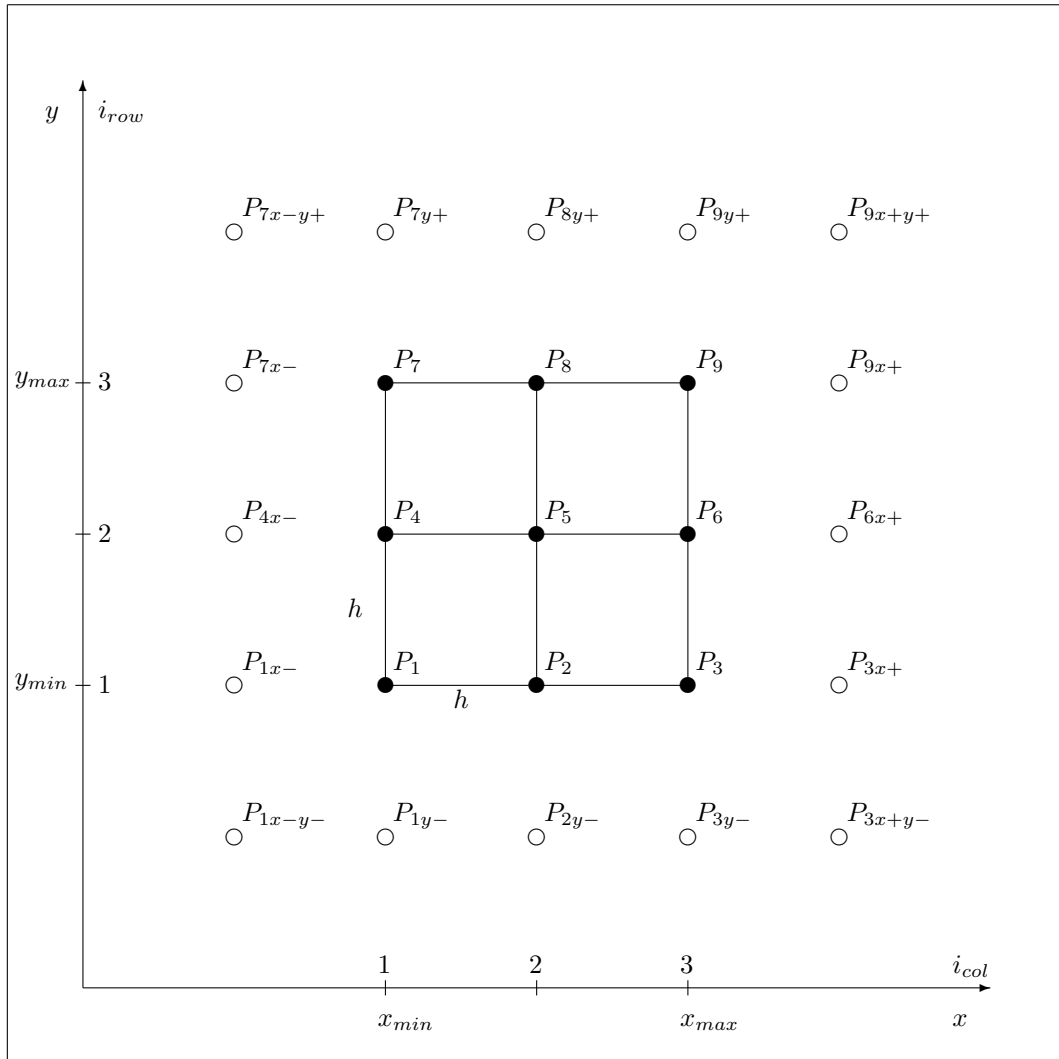


Figure 3: Mesh XY type C

162 **8 Mesh XY - type D**

163 $h_x = h_y = h$

164 gradient V outside a mesh does not exist

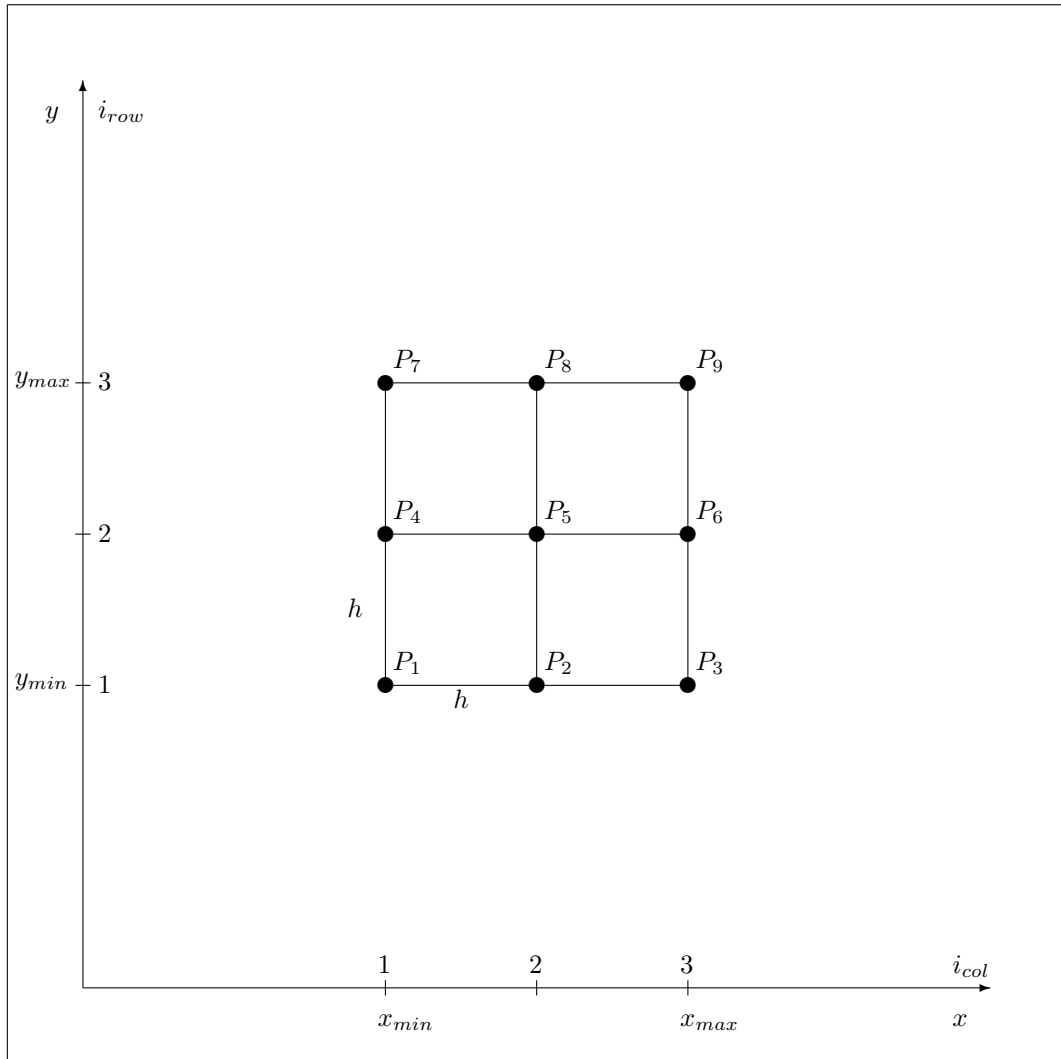


Figure 4: Mesh XY type D

165 **9 Example of A-type mesh in ANSI C**

166 Example of A- type mesh in ANSI C program. The mesh is represented by 2
 167 dimensional array of double precision numbers. Rows and columns in mesh
 168 are numbered from 1 (this was my choice) instead of default 0 (as usual in C
 169 language). This choice nas pros and cons. Is is easier to calculate mesh size
 170 (size_row * size_col). Access to each node can be also more intuitive, but logic
 171 in each library function must contain this shift between node ordering styles.

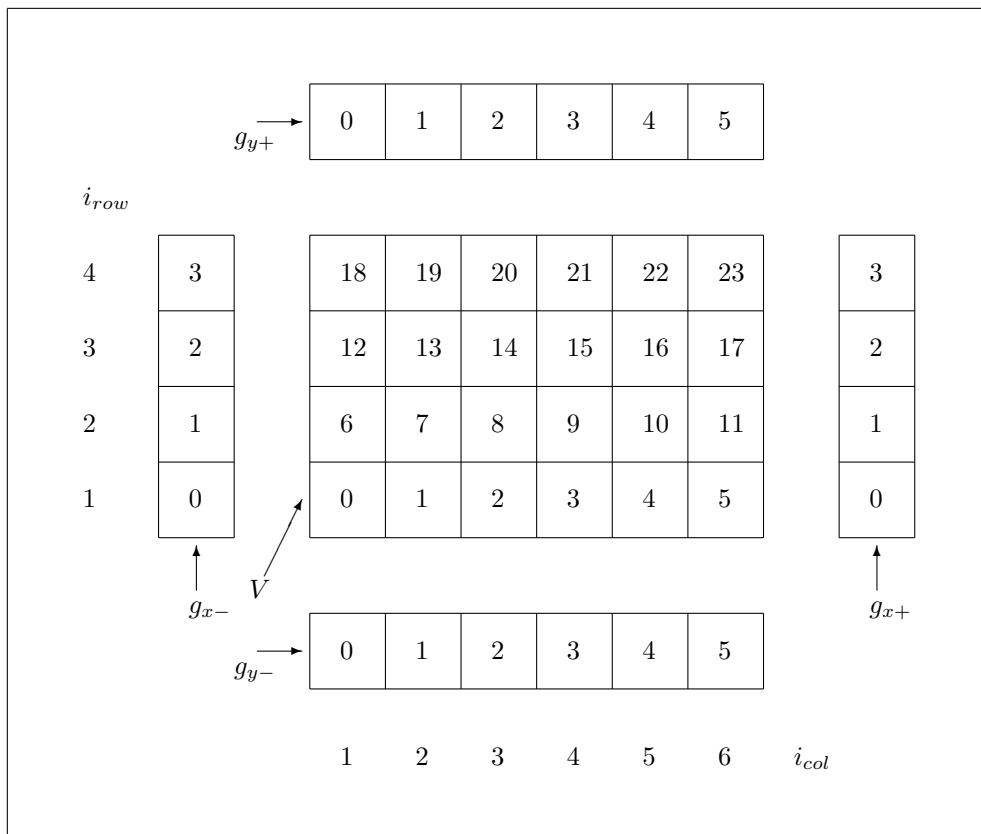


Figure 5: ANSI C - mesh XY type A

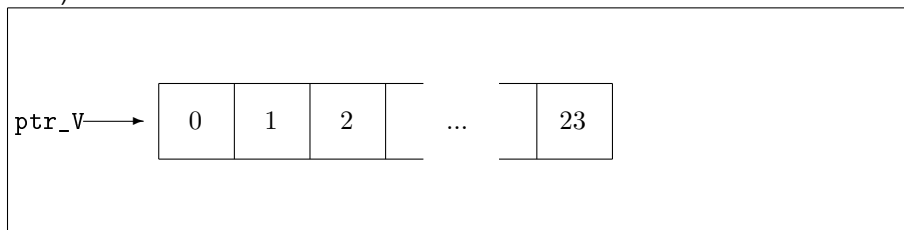
- 172 • $g_{x-} \equiv \text{double* ptr_gX_minus}$
- 173 • $g_{x+} \equiv \text{double* ptr_gX_plus}$
- 174 • $g_{y-} \equiv \text{double* ptr_gY_minus}$
- 175 • $g_{y+} \equiv \text{double* ptr_gY_plus}$
- 176 • $V \equiv \text{double* ptr_V}$
- 177 • `unsigned int size_row == 4`

```

178     • unsigned int size_col == 6
179     • unsigned int i_row == 1, 2, .., 4
180     • unsigned int i_col == 1,2, .., 6
181     • double h_x == 1.0 [mm]
182     • double h_y == 2.0 [mm]

```

183 The following picture describes analogous version of ptr_V mesh, which
184 can be dynamically allocated on heap by pointer method. The mesh is rep-
185 resented by single block of memory. The numbers of rows and columns are
186 also known, so each node can be also accessed by appropriate index (memory
187 address).



188
189 Each mesh point has its unique index (let's say icp - (index of central
190 point)), which can be determined, if we know indices of row and column (i_row,
191 i_col).

$$icp == (i_row - 1) * size_col + i_col - 1 \quad (9.1)$$

192 For example for each point of a mesh indices of row and column have val-
193 ues:

$$\begin{aligned}
i_row &== 1, 2, \dots, size_row \\
i_col &== 1, 2, \dots, size_col
\end{aligned} \quad (9.2)$$

194 **10 Example of B-type mesh in ANSI C**

195 Example of B- type mesh in ANSI C program. The mesh is analogous to A -
196 type mesh. There are no electric field gradients on mesh borders.

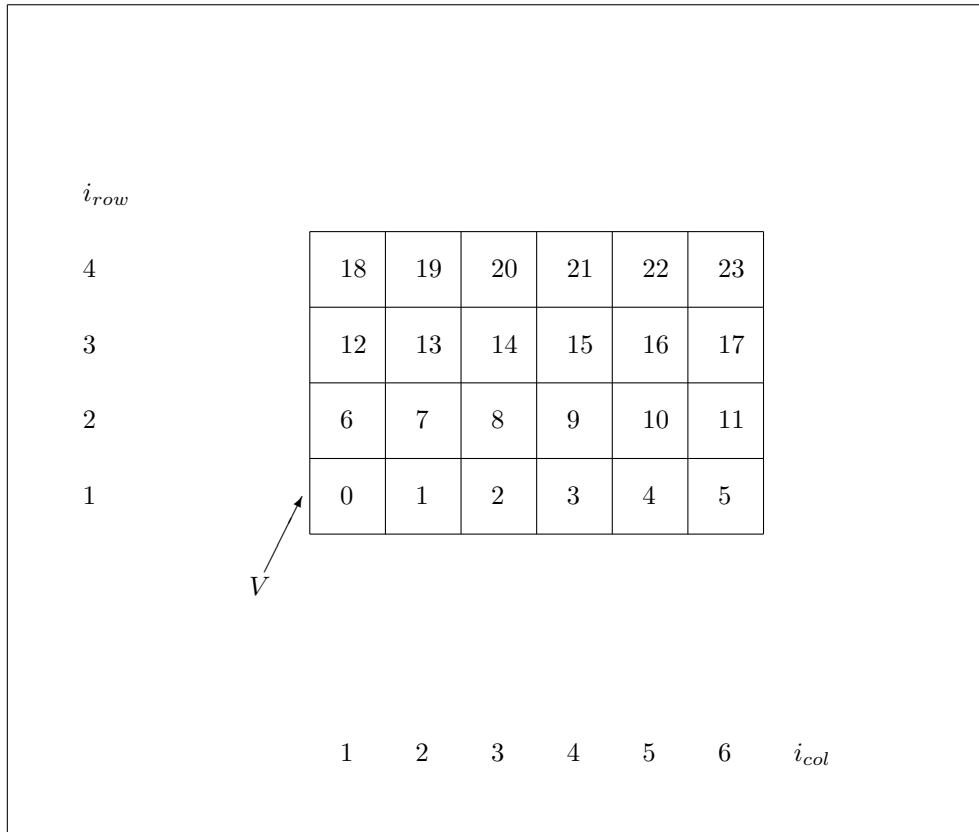


Figure 6: ANSI C - mesh XY type B

- 197 • $V \equiv \text{double* ptr}_V$
- 198 • `unsigned int size_row == 4`
- 199 • `unsigned int size_col == 6`
- 200 • `unsigned int i_row == 1, 2, ..., 4`
- 201 • `unsigned int i_col == 1,2, ..., 6`
- 202 • `double h_x == 1.0 [mm]`
- 203 • `double h_y == 2.0 [mm]`

204 **11 Example of C-type mesh in ANSI C**

205 Example of C- type mesh in ANSI C program. The mesh is analogous to A -
 206 type mesh. Just mesh mesh step $h_x = h_y = h$.

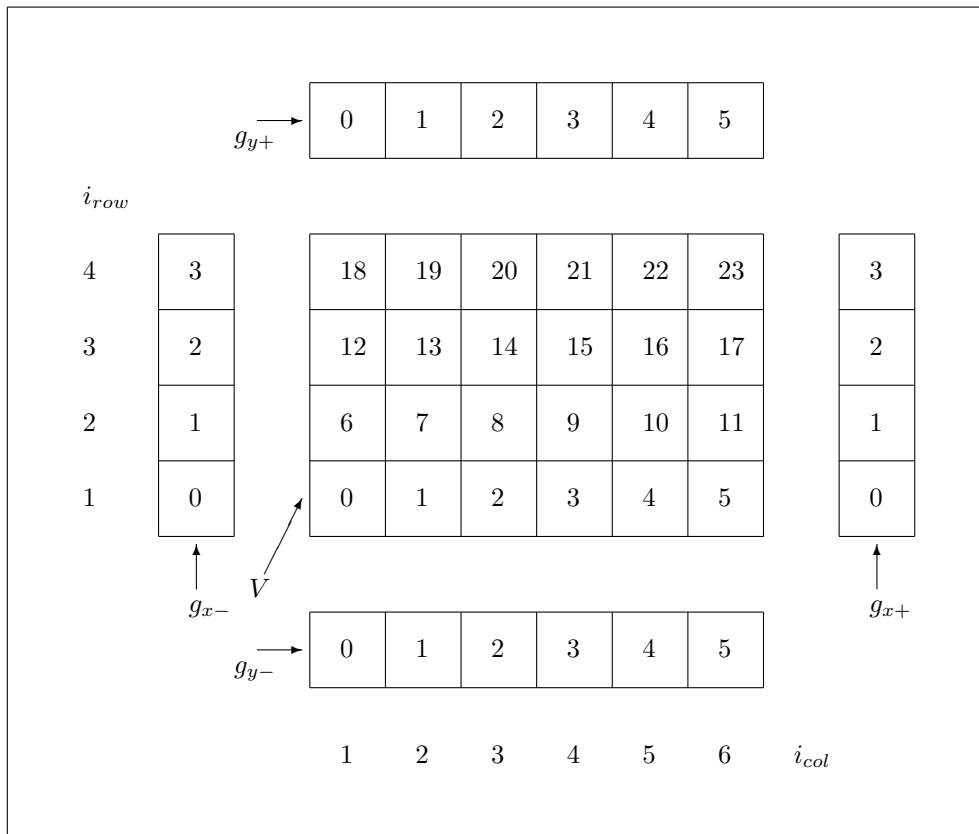


Figure 7: ANSI C - mesh XY type C

- 207 • $g_{x-} \equiv \text{double* ptr_gX_minus}$
- 208 • $g_{x+} \equiv \text{double* ptr_gX_plus}$
- 209 • $g_{y-} \equiv \text{double* ptr_gY_minus}$
- 210 • $g_{y+} \equiv \text{double* ptr_gY_plus}$
- 211 • $V \equiv \text{double* ptr_V}$
- 212 • `unsigned int size_row == 4`
- 213 • `unsigned int size_col == 6`
- 214 • `unsigned int i_row == 1, 2, .., 4`

```
215     • unsigned int i_col == 1,2, .., 6
216     • double h == 1.0 [mm]
```

217 **12 Example of D-type mesh in ANSI C**

218 Example of D- type mesh in ANSI C program. The mesh is analogous to B -
219 type mesh. Just $h_x = h_y = h$.

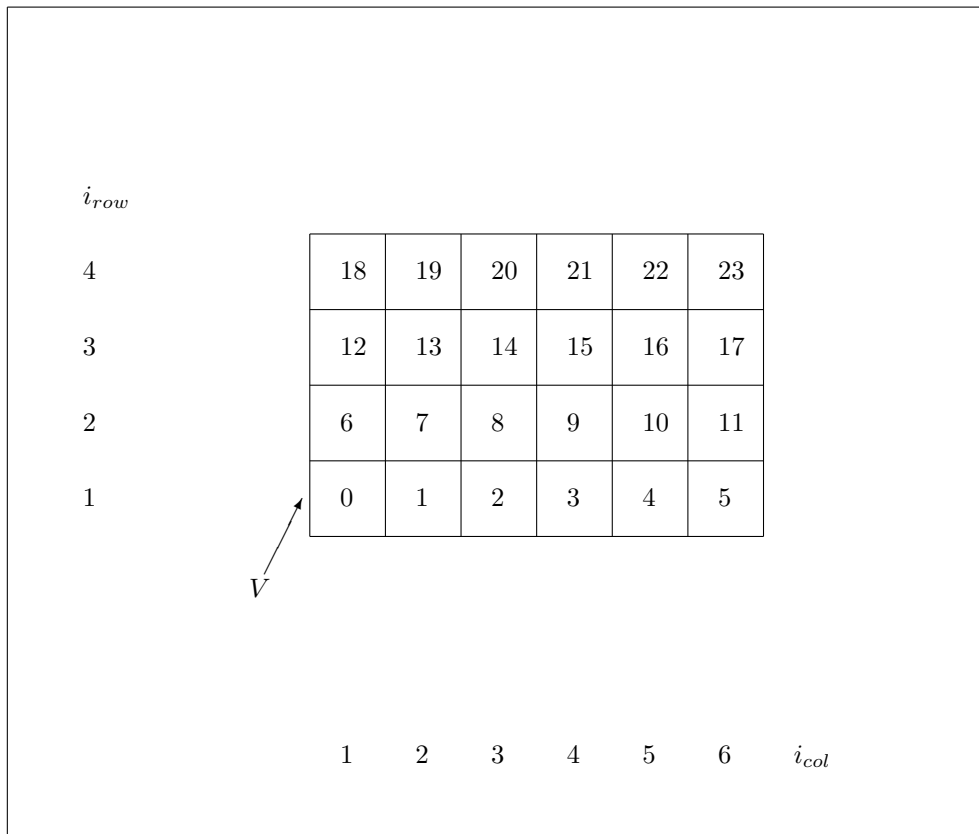


Figure 8: ANSI C - mesh XY type D

- 220 • $V \equiv \text{double* ptr}_V$
- 221 • `unsigned int size_row == 4`
- 222 • `unsigned int size_col == 6`
- 223 • `unsigned int i_row == 1, 2, ..., 4`
- 224 • `unsigned int i_col == 1,2, ..., 6`
- 225 • `double h == 1.0 [mm]`

226 **13 Relaxation formula for node P1**

227 **13.1 Node description**

228 Left, botton corner of mesh XY.

229 **13.2 Calculation of relaxation formula**

230 Laplace equation at node P_1

$$\nabla^2 (V_{(x,y)})_{P_1} = 0 \quad (13.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_1} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_1} = 0 \quad (13.2)$$

231 Approximation of partial derivatives of $V_{(x,y)}$ at node P_1

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_1} \approx \frac{\frac{V_2 - V_1}{h_x} - \frac{V_1 - V_{1y-}}{h_x}}{h_x} = \frac{V_2 - V_1}{h_x^2} - \frac{g_{1x-}}{h_x} \quad (13.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_1} \approx \frac{\frac{V_4 - V_1}{h_y} - \frac{V_1 - V_{1y-}}{h_y}}{h_y} = \frac{V_4 - V_1}{h_y^2} - \frac{g_{1y-}}{h_y} \quad (13.4)$$

232 Let us substitute approximations to Laplace equation.

$$\frac{V_2 - V_1}{h_x^2} - \frac{g_{1x-}}{h_x} + \frac{V_4 - V_1}{h_y^2} - \frac{g_{1y-}}{h_y} = 0 \quad (13.5)$$

233 Let us find V_1

$$V_1 = ? \quad (13.6)$$

$$\frac{V_2 - V_1}{h_x^2} + \frac{V_4 - V_1}{h_y^2} = \frac{g_{1x-}}{h_x} + \frac{g_{1y-}}{h_y} \quad (13.7)$$

234 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (13.8)$$

235 We obtain

$$V_2 h_y^2 - V_1 h_y^2 + V_4 h_x^2 - V_1 h_x^2 = g_{1x-} h_x h_y^2 + g_{1y-} h_x^2 h_y \quad (13.9)$$

$$V_1 (h_x^2 + h_y^2) = V_2 h_y^2 + V_4 h_x^2 - g_{1x-} h_x h_y^2 - g_{1y-} h_x^2 h_y \quad (13.10)$$

236 **13.3 Final forms of relaxation formula**

237 **13.3.1 xyLV_RELAX5_P1_A**

$$\begin{aligned} & h_x \neq h_y \\ & g_{1x-}, g_{1y-} \neq 0 \\ 238 \quad V_1 &= \frac{V_2 h_y^2 + V_4 h_x^2 - g_{1x-} h_x h_y^2 - g_{1y-} h_x^2 h_y}{h_x^2 + h_y^2} \end{aligned} \quad (13.11)$$

239 **13.3.2 xyLV_RELAX5_P1_B**

$$\begin{aligned} & h_x \neq h_y \\ & g_{1x-}, g_{1y-} = 0 \\ V_1 &= \frac{V_2 h_y^2 + V_4 h_x^2}{h_x^2 + h_y^2} \end{aligned} \quad (13.12)$$

240 **13.3.3 xyLV_RELAX5_P1_C**

$$\begin{aligned} & h_x = h_y = h \\ & g_{1x-}, g_{1y-} \neq 0 \\ V_1 &= \frac{V_2 + V_4 - g_{1x-} h - g_{1y-} h}{2} \end{aligned} \quad (13.13)$$

241 **13.3.4 xyLV_RELAX5_P1_D**

$$\begin{aligned} & h_x = h_y = h \\ & g_{1x-}, g_{1y-} = 0 \\ V_1 &= \frac{V_2 + V_4}{2} \end{aligned} \quad (13.14)$$

242 **14 Relaxation formula for node P2**

243 **14.1 Node description**

244 Bottom edge of mesh XY.

245 **14.2 Calculation of relaxation formula**

246 Laplace equation at node P_2

$$\nabla^2 (V_{(x,y)})_{P_2} = 0 \quad (14.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_2} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_2} = 0 \quad (14.2)$$

247 Approximation of partial derivatives of $V_{(x,y)}$ at node P_2

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_2} \approx \frac{\frac{V_3 - V_2}{h_x} - \frac{V_2 - V_1}{h_x}}{h_x} = \frac{V_1 + V_3 - 2V_2}{h_x^2} \quad (14.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_2} \approx \frac{\frac{V_5 - V_2}{h_y} - \frac{V_2 - V_{2y-}}{h_y}}{h_y} = \frac{V_5 - V_2}{h_y^2} - \frac{g_{2y-}}{h_y} \quad (14.4)$$

248 Let us substitute approximations to Laplace equation.

$$\frac{V_1 + V_3 - 2V_2}{h_x^2} + \frac{V_5 - V_2}{h_y^2} - \frac{g_{2y-}}{h_y} = 0 \quad (14.5)$$

249 Let us find V_2

$$V_2 = ? \quad (14.6)$$

$$\frac{V_1 + V_3 - 2V_2}{h_x^2} + \frac{V_5 - V_2}{h_y^2} = \frac{g_{2y-}}{h_y} \quad (14.7)$$

250 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (14.8)$$

251 We obtain

$$V_1 h_y^2 + V_3 h_y^2 - 2V_2 h_y^2 + V_5 h_x^2 = g_{2y-} h_x^2 h_y \quad (14.9)$$

$$V_2 (h_x^2 + h_y^2) = (V_1 + V_3) h_y^2 + V_5 h_x^2 - g_{2y-} h_x^2 h_y \quad (14.10)$$

252 **14.3 Final forms of relaxation formula**

253 **14.3.1 xyLV_RELAX5_P2_A**

$$\begin{aligned} h_x &\neq h_y \\ g_{2y-} &\neq 0 \\ V_2 &= \frac{(V_1 + V_3) h_y^2 + V_5 h_x^2 - g_{2y-} h_x^2 h_y}{h_x^2 + h_y^2} \end{aligned} \quad (14.11)$$

254 **14.3.2 xyLV_RELAX5_P2_B**

$$\begin{aligned} h_x &\neq h_y \\ g_{2y-} &= 0 \\ V_2 &= \frac{(V_1 + V_3) h_y^2 + V_5 h_x^2}{h_x^2 + h_y^2} \end{aligned} \quad (14.12)$$

255 **14.3.3 xyLV_RELAX5_P2_C**

$$\begin{aligned} h_x &= h_y = h \\ g_{2y-} &\neq 0 \\ V_2 &= \frac{V_1 + V_3 + V_5 - g_{2y-} h}{3} \end{aligned} \quad (14.13)$$

256 **14.3.4 xyLV_RELAX5_P2_D**

$$\begin{aligned} h_x &= h_y = h \\ g_{2y-} &= 0 \\ V_2 &= \frac{V_1 + V_3 + V_5}{3} \end{aligned} \quad (14.14)$$

257 **15 Relaxation formula for node P3**

258 **15.1 Node description**

259 Right, botton corner of mesh XY.

260 **15.2 Calculation of relaxation formula**

261 Laplace equation at node P_3

$$\nabla^2 (V_{(x,y)})_{P_3} = 0 \quad (15.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_3} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_3} = 0 \quad (15.2)$$

262 Approximation of partial derivatives of $V_{(x,y)}$ at node P_3

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_3} \approx \frac{V_{3x+} - V_3 - \frac{V_3 - V_2}{h_x}}{h_x} = \frac{g_{3x+}}{h_x} + \frac{V_2 - V_3}{h_x^2} \quad (15.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_3} \approx \frac{\frac{V_6 - V_3}{h_y} - \frac{V_3 - V_{3y-}}{h_y}}{h_y} = \frac{V_6 - V_3}{h_y^2} - \frac{g_{3y-}}{h_y} \quad (15.4)$$

263 Let us substitute approximations to Laplace equation.

$$\frac{g_{3x+}}{h_x} + \frac{V_2 - V_3}{h_x^2} + \frac{V_6 - V_3}{h_y^2} - \frac{g_{3y-}}{h_y} = 0 \quad (15.5)$$

264 Let us find V_3

$$V_3 = ? \quad (15.6)$$

$$\frac{V_2 - V_3}{h_x^2} + \frac{V_6 - V_3}{h_y^2} = \frac{g_{3y-}}{h_y} - \frac{g_{3x+}}{h_x} \quad (15.7)$$

265 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (15.8)$$

266 We obtain

$$V_2 h_y^2 - V_3 h_y^2 + V_6 h_x^2 - V_3 h_x^2 = g_{3y-} h_x^2 h_y - g_{3x+} h_x h_y^2 \quad (15.9)$$

$$V_3 (h_x^2 + h_y^2) = V_2 h_y^2 + V_6 h_x^2 + g_{3x+} h_x h_y^2 - g_{3y-} h_x^2 h_y \quad (15.10)$$

267 **15.3 Final forms of relaxation formula**

268 **15.3.1 xyLV_RELAX5_P3_A**

$$\begin{aligned} & h_x \neq h_y \\ & g_{3x+}, g_{3y-} \neq 0 \\ V_3 &= \frac{V_2 h_y^2 + V_6 h_x^2 + g_{3x+} h_x h_y^2 - g_{3y-} h_x^2 h_y}{h_x^2 + h_y^2} \end{aligned} \quad (15.11)$$

269 **15.3.2 xyLV_RELAX5_P3_B**

$$\begin{aligned} & h_x \neq h_y \\ & g_{3x+}, g_{3y-} = 0 \\ V_3 &= \frac{V_2 h_y^2 + V_6 h_x^2}{h_x^2 + h_y^2} \end{aligned} \quad (15.12)$$

270 **15.3.3 xyLV_RELAX5_P3_C**

$$\begin{aligned} & h_x = h_y = h \\ & g_{3x+}, g_{3y-} \neq 0 \\ V_3 &= \frac{V_2 + V_6 + g_{3x+} h - g_{3y-} h}{2} \end{aligned} \quad (15.13)$$

271 **15.3.4 xyLV_RELAX5_P3_D**

$$\begin{aligned} & h_x = h_y = h \\ & g_{3x+}, g_{3y-} = 0 \\ V_3 &= \frac{V_2 + V_6}{2} \end{aligned} \quad (15.14)$$

272 **16 Relaxation formula for node P4**

273 **16.1 Node description**

274 Left edge of mesh XY.

275 **16.2 Calculation of relaxation formula**

276 Laplace equation at node P_4

$$\nabla^2 (V_{(x,y)})_{P_4} = 0 \quad (16.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_4} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_4} = 0 \quad (16.2)$$

277 Approximation of partial derivatives of $V_{(x,y)}$ at node P_4

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_4} \approx \frac{\frac{V_5 - V_4}{h_x} - \frac{V_4 - V_{4x-}}{h_x}}{h_x} = \frac{V_5 - V_4}{h_x^2} - \frac{g_{4x-}}{h_x} \quad (16.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_4} \approx \frac{\frac{V_7 - V_4}{h_y} - \frac{V_4 - V_1}{h_y}}{h_y} = \frac{V_1 + V_7 - 2V_4}{h_y^2} \quad (16.4)$$

278 Let us substitute approximations to Laplace equation.

$$\frac{V_5 - V_4}{h_x^2} - \frac{g_{4x-}}{h_x} + \frac{V_1 + V_7 - 2V_4}{h_y^2} = 0 \quad (16.5)$$

279 Let us find V_4

$$V_4 = ? \quad (16.6)$$

$$\frac{V_5 - V_4}{h_x^2} + \frac{V_1 + V_7 - 2V_4}{h_y^2} = \frac{g_{4x-}}{h_x} \quad (16.7)$$

280 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (16.8)$$

281 We obtain

$$V_5 h_y^2 - V_4 h_y^2 + V_1 h_x^2 + V_7 h_x^2 - 2V_4 h_x^2 = g_{4x-} h_x h_y^2 \quad (16.9)$$

$$V_4 (2h_x^2 + h_y^2) = (V_1 + V_7) h_x^2 + V_5 h_y^2 - g_{4x-} h_x h_y^2 \quad (16.10)$$

282 **16.3 Final forms of relaxation formula**

283 **16.3.1 xyLV_RELAX5_P4_A**

$$\begin{aligned} h_x &\neq h_y \\ g_{4x-} &\neq 0 \\ V_4 &= \frac{(V_1 + V_7) h_x^2 + V_5 h_y^2 - g_{4x-} h_x h_y^2}{2h_x^2 + h_y^2} \end{aligned} \quad (16.11)$$

284 **16.3.2 xyLV_RELAX5_P4_B**

$$\begin{aligned} h_x &\neq h_y \\ g_{4x-} &= 0 \\ V_2 &= \frac{(V_1 + V_7) h_x^2 + V_5 h_y^2}{2h_x^2 + h_y^2} \end{aligned} \quad (16.12)$$

285 **16.3.3 xyLV_RELAX5_P4_C**

$$\begin{aligned} h_x &= h_y = h \\ g_{4x-} &\neq 0 \\ V_4 &= \frac{V_1 + V_5 + V_7 - g_{4x-} h}{3} \end{aligned} \quad (16.13)$$

286 **16.3.4 xyLV_RELAX5_P4_D**

$$\begin{aligned} h_x &= h_y = h \\ g_{4x-} &= 0 \\ V_4 &= \frac{V_1 + V_5 + V_7}{3} \end{aligned} \quad (16.14)$$

287 **17 Relaxation formula for node P5**

288 **17.1 Node description**

289 Node inside a mesh XY.

290 **17.2 Calculation of relaxation formula**

291 Laplace equation at node P_5

$$\nabla^2 (V_{(x,y)})_{P_5} = 0 \quad (17.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_5} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_5} = 0 \quad (17.2)$$

292 Approximation of partial derivatives of $V_{(x,y)}$ at node P_5

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_5} \approx \frac{\frac{V_6 - V_5}{h_x} - \frac{V_5 - V_4}{h_x}}{h_x} = \frac{V_4 + V_6 - 2V_5}{h_x^2} \quad (17.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_5} \approx \frac{\frac{V_8 - V_5}{h_y} - \frac{V_5 - V_2}{h_y}}{h_y} = \frac{V_2 + V_8 - 2V_5}{h_y^2} \quad (17.4)$$

293 Let us substitute approximations to Laplace equation.

$$\frac{V_4 + V_6 - 2V_5}{h_x^2} + \frac{V_2 + V_8 - 2V_5}{h_y^2} = 0 \quad (17.5)$$

294 Let us find V_5

$$V_5 = ? \quad (17.6)$$

$$\frac{V_4 + V_6 - 2V_5}{h_x^2} + \frac{V_2 + V_8 - 2V_5}{h_y^2} = 0 \quad (17.7)$$

295 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (17.8)$$

296 We obtain

$$V_4 h_y^2 + V_6 h_y^2 - 2V_5 h_y^2 + V_2 h_x^2 + V_8 h_x^2 - 2V_5 h_x^2 = 0 \quad (17.9)$$

$$2V_5 (h_x^2 + h_y^2) = (V_2 + V_8) h_x^2 + (V_4 + V_6) h_y^2 \quad (17.10)$$

297 **17.3 Final forms of relaxation formula**

298 **17.3.1 xyLV_RELAX5_P5_A**

$$h_x \neq h_y$$

299 No gradients g inside mesh are considered.

$$V_5 = \frac{(V_2 + V_8) h_x^2 + (V_4 + V_6) h_y^2}{2(h_x^2 + h_y^2)} \quad (17.11)$$

300 **17.3.2 xyLV_RELAX5_P5_B**

$$h_x \neq h_y$$

301 Relaxation formula is the same as xyLV_RELAX5_P5_A

$$V_5 = \frac{(V_2 + V_8) h_x^2 + (V_4 + V_6) h_y^2}{2(h_x^2 + h_y^2)} \quad (17.12)$$

302 **17.3.3 xyLV_RELAX5_P5_C**

$$h_x = h_y = h$$

303 No gradients g inside mesh are considered.

304 The formula simplifies, so no g and h terms are necessary.

$$V_5 = \frac{V_2 + V_4 + V_6 + V_8}{4} \quad (17.13)$$

305 **17.3.4 xyLV_RELAX5_P5_D**

$$h_x = h_y = h$$

306 The formula also simplifies.

307

308 Relaxation formula is the same as xyLV_RELAX5_P5_C

$$V_5 = \frac{V_2 + V_4 + V_6 + V_8}{4} \quad (17.14)$$

309 **18 Relaxation formula for node P6**

310 **18.1 Node description**

311 Right edge of mesh XY.

312 **18.2 Calculation of relaxation formula**

313 Laplace equation at node P_6

$$\nabla^2 (V_{(x,y)})_{P_6} = 0 \quad (18.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_6} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_6} = 0 \quad (18.2)$$

314 Approximation of partial derivatives of $V_{(x,y)}$ at node P_6

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_6} \approx \frac{V_{6x+} - V_6}{h_x} - \frac{V_6 - V_5}{h_x} = \frac{g_{6x+}}{h_x} + \frac{V_5 - V_6}{h_x^2} \quad (18.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_6} \approx \frac{V_9 - V_6}{h_y} - \frac{V_6 - V_3}{h_y} = \frac{V_3 + V_9 - 2V_6}{h_y^2} \quad (18.4)$$

315 Let us substitute approximations to Laplace equation.

$$\frac{g_{6x+}}{h_x} + \frac{V_5 - V_6}{h_x^2} + \frac{V_3 + V_9 - 2V_6}{h_y^2} = 0 \quad (18.5)$$

316 Let us find V_6

$$V_6 = ? \quad (18.6)$$

$$\frac{V_5 - V_6}{h_x^2} + \frac{V_3 + V_9 - 2V_6}{h_y^2} = -\frac{g_{6x+}}{h_x} \quad (18.7)$$

317 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (18.8)$$

318 We obtain

$$V_5 h_y^2 - V_6 h_y^2 + V_3 h_x^2 + V_9 h_x^2 - 2V_6 h_x^2 = -g_{6x+} h_x h_y^2 \quad (18.9)$$

$$V_6 (2h_x^2 + h_y^2) = (V_3 + V_9) h_x^2 + V_5 h_y^2 + g_{6x+} h_x h_y^2 \quad (18.10)$$

319 **18.3 Final forms of relaxation formula**

320 **18.3.1 xyLV_RELAX5_P6_A**

$$\begin{aligned} h_x &\neq h_y \\ g_{6x+} &\neq 0 \\ V_6 &= \frac{(V_3 + V_9) h_x^2 + V_5 h_y^2 + g_{6x+} h_x h_y^2}{2h_x^2 + h_y^2} \end{aligned} \quad (18.11)$$

321 **18.3.2 xyLV_RELAX5_P6_B**

$$\begin{aligned} h_x &\neq h_y \\ g_{6x+} &= 0 \\ V_6 &= \frac{(V_3 + V_9) h_x^2 + V_5 h_y^2}{2h_x^2 + h_y^2} \end{aligned} \quad (18.12)$$

322 **18.3.3 xyLV_RELAX5_P6_C**

$$\begin{aligned} h_x &= h_y = h \\ g_{6x+} &\neq 0 \\ V_6 &= \frac{V_3 + V_5 + V_9 + g_{6x+} h}{3} \end{aligned} \quad (18.13)$$

323 **18.3.4 xyLV_RELAX5_P6_D**

$$\begin{aligned} h_x &= h_y = h \\ g_{6x+} &= 0 \\ V_6 &= \frac{V_3 + V_5 + V_9}{3} \end{aligned} \quad (18.14)$$

324 **19 Relaxation formula for node P7**

325 **19.1 Node description**

326 Left, upper corner of mesh XY.

327 **19.2 Calculation of relaxation formula**

328 Laplace equation at node P_7

$$\nabla^2 (V_{(x,y)})_{P_7} = 0 \quad (19.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_7} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_7} = 0 \quad (19.2)$$

329 Approximation of partial derivatives of $V_{(x,y)}$ at node P_7

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_7} \approx \frac{\frac{V_8 - V_7}{h_x} - \frac{V_7 - V_{7x-}}{h_x}}{h_x} = \frac{V_8 - V_7}{h_x^2} - \frac{g_{7x-}}{h_x} \quad (19.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_7} \approx \frac{\frac{V_{7y+} - V_7}{h_y} - \frac{V_7 - V_4}{h_y}}{h_y} = \frac{V_4 - V_7}{h_y^2} + \frac{g_{7y+}}{h_y} \quad (19.4)$$

330 Let us substitute approximations to Laplace equation.

$$\frac{V_8 - V_7}{h_x^2} - \frac{g_{7x-}}{h_x} + \frac{V_4 - V_7}{h_y^2} + \frac{g_{7y+}}{h_y} = 0 \quad (19.5)$$

331 Let us find V_7

$$V_7 = ? \quad (19.6)$$

$$\frac{V_8 - V_7}{h_x^2} + \frac{V_4 - V_7}{h_y^2} = \frac{g_{7x-}}{h_x} - \frac{g_{7y+}}{h_y} \quad (19.7)$$

332 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (19.8)$$

333 We obtain

$$V_8 h_y^2 - V_7 h_y^2 + V_4 h_x^2 - V_7 h_x^2 = g_{7x-} h_x h_y^2 - g_{7y+} h_x^2 h_y \quad (19.9)$$

$$V_7 (h_x^2 + h_y^2) = V_4 h_x^2 + V_8 h_y^2 - g_{7x-} h_x h_y^2 - g_{7y+} h_x^2 h_y \quad (19.10)$$

334 **19.3 Final forms of relaxation formula**

335 **19.3.1 xyLV_RELAX5_P7_A**

$$\begin{aligned} h_x &\neq h_y \\ g_{7x-}, g_{7y+} &\neq 0 \\ V_7 &= \frac{V_4 h_x^2 + V_8 h_y^2 - g_{7x-} h_x h_y^2 + g_{7y+} h_x^2 h_y}{(h_x^2 + h_y^2)} \end{aligned} \quad (19.11)$$

336 **19.3.2 xyLV_RELAX5_P7_B**

$$\begin{aligned} h_x &\neq h_y \\ g_{7x-}, g_{7y+} &= 0 \\ V_7 &= \frac{V_4 h_x^2 + V_8 h_y^2}{h_x^2 + h_y^2} \end{aligned} \quad (19.12)$$

337 **19.3.3 xyLV_RELAX5_P7_C**

$$\begin{aligned} h_x &= h_y = h \\ g_{7x-}, g_{7y+} &\neq 0 \\ V_7 &= \frac{V_4 + V_8 - g_{7x-} h + g_{7y+} h}{2} \end{aligned} \quad (19.13)$$

338 **19.3.4 xyLV_RELAX5_P7_D**

$$\begin{aligned} h_x &= h_y = h \\ g_{7x-}, g_{7y+} &= 0 \\ V_7 &= \frac{V_4 + V_8}{2} \end{aligned} \quad (19.14)$$

339 **20 Relaxation formula for node P8**

340 **20.1 Node description**

341 Upper edge of mesh XY.

342 **20.2 Calculation of relaxation formula**

343 Laplace equation at node P_8

$$\nabla^2 (V_{(x,y)})_{P_8} = 0 \quad (20.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_8} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_8} = 0 \quad (20.2)$$

344 Approximation of partial derivatives of $V_{(x,y)}$ at node P_8

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_8} \approx \frac{\frac{V_9 - V_8}{h_x} - \frac{V_8 - V_7}{h_x}}{h_x} = \frac{V_7 + V_9 - 2V_8}{h_x^2} \quad (20.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_8} \approx \frac{\frac{V_{8y+} - V_8}{h_y} - \frac{V_8 - V_5}{h_y}}{h_y} = \frac{V_5 - V_8}{h_y^2} + \frac{g_{8y+}}{h_y} \quad (20.4)$$

345 Let us substitute approximations to Laplace equation.

$$\frac{V_7 + V_9 - 2V_8}{h_x^2} + \frac{V_5 - V_8}{h_y^2} + \frac{g_{8y+}}{h_y} = 0 \quad (20.5)$$

346 Let us find V_8

$$V_8 = ? \quad (20.6)$$

$$\frac{V_7 + V_9 - 2V_8}{h_x^2} + \frac{V_5 - V_8}{h_y^2} = -\frac{g_{8y+}}{h_y} \quad (20.7)$$

347 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (20.8)$$

348 We obtain

$$V_7 h_y^2 + V_9 h_y^2 - 2V_8 h_y^2 + V_5 h_x^2 - V_8 h_x^2 = -g_{8y+} h_x^2 h_y \quad (20.9)$$

$$V_8 (h_x^2 + 2h_y^2) = (V_7 + V_9) h_y^2 + V_5 h_x^2 + g_{8y+} h_x^2 h_y \quad (20.10)$$

349 **20.3 Final forms of relaxation formula**

350 **20.3.1 xyLV_RELAX5_P8_A**

$$\begin{aligned} h_x &\neq h_y \\ g_{8y+} &\neq 0 \\ V_8 &= \frac{V_5 h_x^2 + (V_7 + V_9) h_y^2 + g_{8y+} h_x^2 h_y}{h_x^2 + 2h_y^2} \end{aligned} \quad (20.11)$$

351 **20.3.2 xyLV_RELAX5_P8_B**

$$\begin{aligned} h_x &\neq h_y \\ g_{8y+} &= 0 \\ V_8 &= \frac{V_5 h_x^2 + (V_7 + V_9) h_y^2}{h_x^2 + 2h_y^2} \end{aligned} \quad (20.12)$$

352 **20.3.3 xyLV_RELAX5_P8_C**

$$\begin{aligned} h_x &= h_y = h \\ g_{8y+} &\neq 0 \\ V_8 &= \frac{V_5 + V_7 + V_9 + g_{8y+} h}{3} \end{aligned} \quad (20.13)$$

353 **20.3.4 xyLV_RELAX5_P8_D**

$$\begin{aligned} h_x &= h_y = h \\ g_{8y+} &= 0 \\ V_8 &= \frac{V_5 + V_7 + V_9}{3} \end{aligned} \quad (20.14)$$

354 **21 Relaxation formula for node P9**

355 **21.1 Node description**

356 Right, upper corner of mesh XY.

357 **21.2 Calculation of relaxation formula**

358 Laplace equation at node P_9

$$\nabla^2 (V_{(x,y)})_{P_9} = 0 \quad (21.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_9} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_9} = 0 \quad (21.2)$$

359 Approximation of partial derivatives of $V_{(x,y)}$ at node P_9

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_9} \approx \frac{\frac{V_{9x+} - V_9}{h_x} - \frac{V_9 - V_8}{h_x}}{h_x} = \frac{V_8 - V_9}{h_x^2} + \frac{g_{9x+}}{h_x} \quad (21.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_9} \approx \frac{\frac{V_{9y+} - V_9}{h_y} - \frac{V_9 - V_6}{h_y}}{h_y} = \frac{V_6 - V_9}{h_y^2} + \frac{g_{9y+}}{h_y} \quad (21.4)$$

360 Let us substitute approximations to Laplace equation.

$$\frac{V_8 - V_9}{h_x^2} + \frac{g_{9x+}}{h_x} + \frac{V_6 - V_9}{h_y^2} + \frac{g_{9y+}}{h_y} = 0 \quad (21.5)$$

361 Let us find V_9

$$V_9 = ? \quad (21.6)$$

$$\frac{V_8 - V_9}{h_x^2} + \frac{V_8 - V_9}{h_x^2} = -\frac{g_{9x+}}{h_x} - \frac{g_{9y+}}{h_y} \quad (21.7)$$

362 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (21.8)$$

363 We obtain

$$V_8 h_y^2 - V_9 h_y^2 + V_6 h_x^2 - V_9 h_x^2 = -g_{9x+} h_x h_y^2 - g_{9y+} h_x^2 h_y \quad (21.9)$$

$$V_9 (h_x^2 + h_y^2) = V_6 h_x^2 + V_8 h_y^2 + g_{9x+} h_x h_y^2 + g_{9y+} h_x^2 h_y \quad (21.10)$$

364 **21.3 Final forms of relaxation formula**

365 **21.3.1 xyLV_RELAX5_P9_A**

$$\begin{aligned} & h_x \neq h_y \\ & g_{9x+}, g_{9y+} \neq 0 \\ V_9 &= \frac{V_6 h_x^2 + V_8 h_y^2 + g_{9x+} h_x h_y^2 + g_{9y+} h_x^2 h_y}{h_x^2 + h_y^2} \end{aligned} \quad (21.11)$$

366 **21.3.2 xyLV_RELAX5_P9_B**

$$\begin{aligned} & h_x \neq h_y \\ & g_{9x+}, g_{9y+} = 0 \\ V_9 &= \frac{V_6 h_x^2 + V_8 h_y^2}{h_x^2 + h_y^2} \end{aligned} \quad (21.12)$$

367 **21.3.3 xyLV_RELAX5_P9_C**

$$\begin{aligned} & h_x = h_y = h \\ & g_{9x+}, g_{9y+} \neq 0 \\ V_9 &= \frac{V_6 + V_8 + g_{9x+} h + g_{9y+} h}{2} \end{aligned} \quad (21.13)$$

368 **21.3.4 xyLV_RELAX5_P9_D**

$$\begin{aligned} & h_x = h_y = h \\ & g_{9x+}, g_{9y+} = 0 \\ V_9 &= \frac{V_6 + V_8}{2} \end{aligned} \quad (21.14)$$

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