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Characterizations of probability distributions by regressions of non-adjacent generalized order statistics and discrete record values

ABSTRACT

In the dissertation, we study the problem of the unique identification of probability distributions by the knowledge of single regression function of various models of ordered statisticaldata. In particular we provide a new approach to this characterization problem based on the Markov property of these models. In the case of continuous distributions, this problem is considered for the model of generalized order statistics which includes as special cases many modelsof ordered random variables, such as ordinary order statistics, record values, *k*-th record values,sequential order statistics and progressively type II censored order statistics, among others. The main regression function is then given by

$$E\left(h\left(X_*^{(r+\ell)}\right)|X_*^{(r)} = x\right) = \xi(x),$$

where *h* is a known strictly increasing function and $X_*^{(r)}$, $X_*^{(r+\ell)}$ denote the *r*-th and $(r+\ell)$-th generalized order statistics based on an arbitrary continuous distribution *F*, respectively. Whereas, in the case of discrete probability distributions we analyse an analogous characterization problem by regression functions of discrete record values.

In Chapter 1 we recall the basic definitions and properties of the considered models of ordered random variables which are needed in our considerations. We discuss in detail the characterization problem of probability distributions by the knowledge of single regression of thesemodels and present an outline of the new approach to this characterization problem based on the Markov property of random variables. Using this property, a specific recurrence structure of the regressions is shown. In particular, this property allows to replace the regression of non-adjacent generalized order statistics by appropriately modified regression of adjacent variables in this model. In this chapter we also study significant properties of regression functions of ordered random variables.

In Chapter 2, we consider the uniqueness of characterization of absolutely continuous distributions with continuous density function by the single regression of non-adjacent generalized order statistics. Utilizing their Markov property we show that the characterization is unique if and only if the corresponding system of $\ell - 1$ differential equations with some initial conditions has the unique solution. In particular, for $\ell = 2$ the regression ξ of generalized order statistics with parameters γ_{r+1} and γ_{r+2} determines *F* uniquely if and only if the ordinary differential equation

$$y' = \frac{\gamma_{r+2}}{\gamma_{r+1}} \frac{y - h(x)}{\xi(x) - y} \xi'(x)$$

has the unique solution φ such that

$$h(x) < \varphi(x) < \xi(x), \quad x \in (\alpha,\beta),$$
$$\int_\alpha^x \frac{\xi'(t)}{\xi(t)-\varphi(t)}\,dt < \infty, \quad x \in (\alpha,\beta),$$

$$\int_\alpha^\beta \frac{\xi'(t)}{\xi(t)-\varphi(t)}\,dt = \infty$$

and

$$\lim_{x\to\beta^-} \xi(x)\exp\left(-\int_\alpha^x \frac{\xi'(t)}{\xi(t)-\varphi(t)}\,dt\right) = 0.$$

Moreover, then the underlying distribution function F is given by the inversion formula

$$\mathrm{F(x)} = 1-\exp\left(-\frac{1}{\gamma_{r+1}}\int_\alpha^x \frac{\xi'(t)}{\xi(t)-\varphi(t)}\,dt\right).$$

This result is new even in the particular case of ordinary order statistics. Then our approach we apply to derive new characterizations of absolutely continuous distributions and provide a new proof of characterization of power, exponential, and Pareto distributions by linearity of corresponding regressions. We give an exemplary characterization of a particular beta distribution, and we show that normal, Gompertz, and Weibull with a shape parameter $\delta > 1$ distributionsare uniquely characterized by their corresponding regressions of generalized order statistics with $\ell=2$.

In the next chapter of the dissertation, we extend new approach to the characterization problem to arbitrary continuous distributions possibly without density function. In this general case, we prove that the uniqueness of characterization of the underlying distribution holds if and only if the appropriate system of integral equations with non-classical initial conditions has unique solution. Using this result we provide new characterizations of continuous distributions. Especially, it is shown that gamma, Gumbel, and logistic distributions are uniquely characterized by their corresponding regressions of generalized order statistics with $\ell=2$.

In Chapter 4 we consider an analogous problem of the unique identification of discrete probability distributions by regression conditions involving weak record values of the type

$$E(h(W_{r+\ell})|W_r = x) = \xi(x),$$

where W_r, $W_{r+\ell}$ denote the r-th and $(r+\ell)$-th week record value from some discrete probability distribution, respectively. For $\ell \geq 2$, we show that the uniqueness of the characterization of the underlying distribution in this case is equivalent to the uniqueness of solution to a corresponding difference equation ($\ell=2$) or an appropriate system of difference equations ($\ell > 2$) with somenon-classical conditions. This new criterion of uniqueness is then applied to obtain known as well as new characterizations of discrete distributions. In particular, we show that the Poisson and negative binomial distributions are uniquely characterized by their corresponding regression functions of weak record values with $\ell = 2$.

The results of the dissertation were published in the following articles:

- M. Bieniek, K. Maciąg, *Uniqueness of characterization of absolutely continuous distributions by regressions of generalized order statistics*, AStA Advances in Statistical Ana-lysis, 102(3):359–380, 2018.

- M. Bieniek, K. Maciąg, *On the unique characterization of continuous distributions by single regression of non–adjacent generalized order statistics*, Journal of Integral Equationsand Applications, 30(4):491–519, 2018.

- M. Bieniek, K. Maciąg, *On the problem of the unique characterization of discrete distributions by single regression of weak records*, Statistics, 52(3):533–551, 2018.