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The summary of the PhD thesis

Holomorphic extension of Hadamard product of holomorphic functions

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The Hadamard product $f*g$ of functions f and g holomorphic in the unit disk $\mathbb{D} := \mathbb{D}(0,1)$, where $\mathbb{D}(a,r) := \{z \in \mathbb{C} : |z-a| < r\}$ for $a \in \mathbb{C}$ and $r \geqslant 0$, was defined in the 19th century by Hadamard. In the case where functions f and g are holomorphic in a certain neighborhood of the origin it the product is defined by the formula

$$\mathbb{D}(0, R_{f,g}) \ni z \mapsto f * g(z) := \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)g^{(n)}(0)}{(n!)^2} z^n,$$

where $\mathrm{R}_{f,g}$ is the radius of convergence or the power series. In some cases the Hadamard product $f*g$ can be holomorphically extended outside the disk $\mathbb{D}(0, R_{f,g})$. In natural way there arised the problem of determining a universal domain $\Omega \subset \mathbb{C}$, to which one can extend holomorphically a function $f*g$ for all functions $f \in \mathrm{Hol}(A)$ and $g \in \mathrm{Hol}(B)$, where $A, B \subset \mathbb{C}$ are domains containing the origin and the symbol $\mathrm{Hol}(\Omega)$ stands for the class of all holomorphic functions in a domain Ω. Set $\rho(D) := \sup(\{r \geqslant 0 : \mathbb{D}(0,r) \subset D\})$ for a set $D \subset \mathbb{C}$ containing the origin. For given domains $A, B \subset \mathbb{C}$ we define the class $\mathcal{H}_1(A,B)$ of all domains $\Omega \subset \mathbb{C}$ to which a function $(f*g)|_{\mathbb{D}(0,\rho(A)\rho(B))}$ has holomorphic extension. One can show in a simple way that for every $\Omega \in \mathcal{H}_1(A,B)$,

$$\Omega \subset A * B := \hat{\mathbb{C}} \setminus (\hat{\mathbb{C}} \setminus A \cdot \hat{\mathbb{C}} \setminus B),$$

where $\cdot$ denotes the algebraic product of sets. Operation $*$ is called the star product. The problem of holomorphic extensibility of the Hadamard product was solved in 1992 by Müller. Using the fact that the Hadamard product can be represented in the form of Parseval integral, he proved that the maximal in the sense of inclusion domain in $\mathcal{H}_1(A,B)$ is the connected component of $A*B$ containing the origin. The following concept of generalization of the Hadamard product became the motivation for writing this dissertation thesis. For all open sets $A, B \subset \mathbb{C}$ let $\mathcal{H}_2(A,B)$ stand for the class of all open sets $\Omega \subset \mathbb{C}$ for which there exists an operator $T : \mathrm{Hol}(A) \times \mathrm{Hol}(B) \to \mathrm{Hol}(\Omega)$ satisfying the following conditions:

(i) $T(f,g) = f*g$ for all polynomials f and g;

(ii) For all sequences $\mathbb{N} \ni n \mapsto f_n \in \mathrm{Hol}(A)$ and $\mathbb{N} \ni n \mapsto g_n \in \mathrm{Hol}(B)$ and all $f \in \mathrm{Hol}(A)$ and $g \in \mathrm{Hol}(B)$, if $f_n \xrightarrow{\text{ucc}} f$ in A and $g_n \xrightarrow{\text{ucc}} g$ in B, as $n \to +\infty$, then $T(f_n, g_n) \xrightarrow{\text{ucc}} T(f,g)$ in Ω, as $n \to +\infty$,

where the symbol $\xrightarrow{\text{ucc}}$ denotes uniform convergence on compact sets.

The first chapter is entirely devoted to the star product and its properties. Section 1.1 contains the basic properties of star product that follow directly from the definition, such as connectivity, commutativity or monotonicity. In section 1.2 an alternative characterization of the star product and its applications are presented. Section 1.3 contains theorems about the star product, which can be deduced from the special properties of mappings $\hat{\mathbb{C}} \ni z \mapsto T_p(z) := pz$ for $p \in \mathbb{C} \setminus \{0\}$. Sections 1.4 and 1.5 concern the topological and geometrical properties of the star product, respectively. In section 1.6 there are presented the characterizations of star product in the case, where the sets A and B are spirallike domains with respect to the origin.

In the second chapter the λ-Hadamard product H_λ was defined, where $\lambda : \mathbb{Z} \to \mathbb{C}$ is a given sequence satisfying the following condition

$$\limsup_{n\to+\infty} \sqrt[n]{|\lambda_n|} = 0 \quad \text{and} \quad \limsup_{n\to+\infty} \sqrt[n]{|\lambda_{-n}|} = 0;$$

cf. Definition 2.2. In the case, where $\lambda_0 = 1$ and $\lambda_k = 0$ for $k \in \mathbb{Z}\setminus\{0\}$, the operator H_λ coincides with the Hadamard product operator $*$. Therefore the operator H_λ is a natural generalization of the operator $*$. Replacing the Hadamard product $*$ by λ-Hadamard product H_λ one can define in natural manner the counterparts $\mathcal{H}_1^\lambda(A, B)$ and $\mathcal{H}_2^\lambda(A, B)$ of the classes $\mathcal{H}_1(A, B)$ and $\mathcal{H}_2(A, B)$, respectively. Section 2.1 contains the definition and basic properties of the λ-Hadamard product. The main result of the dissertation thesis is Theorem 2.11, which implies in particular that $A * B \in \mathcal{H}_2^\lambda(A, B)$. Moreover, from Theorem 2.11 it follows that the connected component of the set $A*B$ containing the origin belongs to the class $\mathcal{H}_1^\lambda(A, B)$; cf. Corollary 2.12. The Parseval integral formula of H_λ described in Theorem 2.10 plays a key role in the proof of Theorem 2.11. All the results together with their proofs were presented in Section 2.3. Section 2.2 provides a few auxiliary lemmas useful in the proof of Theorem 2.11.

The third chapter contains examples which illustrate the considerations from previous two chapters. Examples in Section 3.1 deal with applications of various methods of determining the star product of sets. Section 3.2 indicates the possible applications of results from the second chapter.

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