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The summary of the PhD thesis

The Schwarz type inequalities for harmonic functions of the unit disc into itself normalized on the boundary

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Let Ω denote a domain in the complex plane $E(\mathbb{C}) := (\mathbb{C}, \rho_e)$, where ρ_e is the standard metric, with a boundary $\Gamma \neq \emptyset$. Setting an arbitrarily non-empty set $A \subset \Gamma$ and a function $f : A \to \mathbb{C}$ we can pose an interesting problem of studying the class $\mathcal{D}_A(\Omega, f)$ consisting of all continuous functions $F : \Omega \cup A \to \mathbb{C}$, which coincide with the function f on the set A and are harmonics in Ω, i.e. functions F are twice continuously differentiable in Ω and satisfy the Laplace differential equation. In particular, if $A = \Gamma$ then the problem of studying the class $\mathcal{D}_\Gamma(\Omega, f)$ reduces to the classical Dirichlet problem for the domain Ω with boundary function $f : \Gamma \to \mathbb{C}$. In the case of the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ its boundary is the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and the unique solution to the Dirichlet problem for $\mathbb{D}$ with continuous boundary function $f : \mathbb{T} \to \mathbb{C}$ can be determined by using the Poisson integral $F = \mathrm{P}[f]$.

This dissertation is relevant largely to the problem posed above for the class $\mathcal{D}_A(\mathbb{D}, f) \cap \mathcal{A}$, where A is a finite subset of the circle $\mathbb{T}$, $\mathcal{A}$ is a class of all bijections of the set $\mathbb{D} \cup A$ on itself and $f(z) = z$ for $z \in A$. The results obtained in this area are presented in Chapter 4 and depend on the real parameter δ characterizing the distribution of points of the set A on the circle $\mathbb{T}$. They are a consequence of the results obtained in Chapter 3 and will be discussed in details later on.

The problem of studying the class $\mathcal{D}_A(\mathbb{D}, f) \cap \mathcal{A}$ is not easy. Pioneering results in this area were obtained in 2015 by D. Partyka and J. Zając for the set $A := \{e^{2\pi i k/3} : k \in \{0, 1, 2\}\}$. They estimated the modulus $|F(z)|$ for $F \in \mathcal{D}_A(\mathbb{D}, f) \cap \mathcal{A}$ and $z \in \mathbb{D}$. They used a method based on precise estimation of this modulus in a wider class of harmonic functions $F : \mathbb{D} \to \mathbb{D}$ satisfying the following sectorial condition: for every $k \in \{1, 2, 3\}$ and for almost every $u \in T_k := \{e^{2\pi i t/3} : t \in [k-1; k]\}$ the radial limit of a function F in a point u belongs to the convex hull, spanned by the origin and the arc T_k. These results were the inspiration for writing this dissertation. In 2018 A. Futa and D. Partyka considered a more general case in which three arcs were replaced by a finite sequence $T_1, T_2, \ldots, T_n$ of closed arcs contained in $\mathbb{T}$ with positive length, total length 2π and covering $\mathbb{T}$. This sequence was named *the partition of a circle* $\mathbb{T}$, and the sectorial condition corresponding to these arcs was called *the sectorial boundary normalization associated with the partition* $T_1, T_2, \ldots, T_n$ *of the circle* $\mathbb{T}$; cf. Definitions 1.1 and 1.2. It should be mentioned that all the results obtained in 2018 by A. Futa and D. Partyka are valid under the default assumption that all arcs of a partition of the circle $\mathbb{T}$ do not exceed half the length of the circle. This restriction is removed in the dissertation, and so the results obtained in it are valid for all partitions.

The leading issue of this dissertation is to estimate the modulus of harmonic functions $F : \mathbb{D} \to \mathbb{D}$ with sectorial normalization, which leads to the Schwarz type inequality for the class of such functions. The results obtained in this scope are present in Chapter 3, and their proofs are based on supporting facts gathered in Chapter 2.

To be more specific, the structure of this thesis consists of four chapters. Chapter 1 contains basic concepts and symbols used in the dissertation as well as general assumptions throughout the text. Then the considered classes of functions were defined, in particular the class of harmonic functions. Basic information on the Poisson integral in the unit disc $\mathbb{D}$ were also reminded. The title Schwarz type inequality was defined and examples of such inequalities were indicated for selected classes of functions. Then *the sectorial boundary normalization*, crucial in further considerations, was introduced.

Chapter 2 provides auxiliary results. They concern the harmonic measure and its properties. The definition of *a strongly extreme point of a set* and related to this definition Lemma 2.8 are of special interest here. Moreover, the class of harmonic mappings satisfying the sectorial normalization was characterized and several useful properties of this class were proved.

The considerations in Chapter 3 are based on Theorem 3.1. This is an improved version of the result obtained by A. Futa and D. Partyka in 2018 by droping the previously mentioned restriction on the partitions of $\mathbb{T}$ and adding the necessary and sufficient conditions for the equality in the estimate in Theorem 3.1. It is worth noting that this theorem is a useful tool for studying extreme configurations. Next the considerations focus on partitions of the circle $\mathbb{T}$ on n arcs with any length. Corollary 3.3 provides an estimate of the module $|F(z)|$, which depends on the number n arcs of the partition $T_1, T_2, \ldots, T_n$ of the circle $\mathbb{T}$, half length δ its shortest arc and the value $p(z)$ being the smallest harmonic measure at the point z with respect to these arcs. The estimate of $p(z)$ from Corollary 2.5 was applied to estimate the modulus $|F(z)|$ in radial form, which depends on the parameters n, δ and $|z|$. In both cases the function ρ_n is used which is defined by the formula (3.19). The problem of estimating and determining $\rho_n(\delta)$ is discussed in remaining part of this chapter. The estimate for $\rho_n(\delta)$ is given in Corollary 3.10, from which the Schwarz type inequality for partitions on n arcs, given in Theorem 3.11, was derived. Subsequently, the cases of partitions on three, and then on four arcs, were considered. For this purpose, the value $\rho_3(\delta)$ for $\delta \in (0; \pi/3]$ was determined; cf. Corollary 3.15. This together with Corollary 3.3 led to the Schwarz type inequality for partitions on three arcs, presented in Theorem 3.16. Moreover, by using Theorem 3.1 all extreme cases, i.e. all pairs (F, z), such that the equality in the estimation of the modulus $|F(z)|$ holds, were characterized. In particular, for $\delta = \pi/3$ the estimate of $|F(z)|$ coincides with the result obtained by D. Partyka and J. Zając in 2015. Then the value $\rho_4(\delta)$ for $\delta \in (0; \pi/4]$ was determined; cf. Corollary 3.20. Consequently, the Schwarz type inequality for partitions on four arcs was derived and all possible extreme cases were discussed; cf. Theorem 3.21.

The last chapter deals with applications. More precisely, it contains examples of applications of the results obtained in Chapter 3. In particular, the presented results concern to the Schwarz type inequalities, the variants of Heinz inequality and the co-Lipschitz constant for harmonic diffeomorphisms of the unit disc $\mathbb{D}$ onto itself satisfying classical boundary normalization.

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