Magdalena Jastrzębska
Maria Curie-Skłodowska University
in Lublin

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The summary of the PhD thesis

Boundary characterizations of complex quasiregular mapings of the unit disk onto itself

Author: mgr inż. Magdalena Jastrzębska
Supervisor: dr hab. Dariusz Partyka, prof. KUL

In the 1987 Krzyż characterized the boundary values of quasiconformal mappings of the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ onto itself. To this aim he carried on to the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ the Beurling-Ahlfors quasisymmetry condition for increasing homeomorphisms of the real line $\mathbb{R}$ onto itself. The characterization was described by Theorems 4.1 and 4.2, which became an inspiration for writing this dissertation thesis. The main purpose of the thesis is to generalize the Krzyż results to more general complex quasiregular mappings.

Chapter 1 contains the basic facts on quasiregular and quasiconformal mappings which are defined in real analysis in Euclidean spaces; cf. Sections 1.1 and 1.2. From the point of view of the complex analysis, the case of such two-dimensional mappings in the Euclidean space is particularly important. They can be described in terms of complex analysis as mappings in the complex plane $E(\mathbb{C})$. This issue is discussed in Section 1.3. The complex quasiregular mappings, defined in this manner, naturally generalize holomorphic functions, and therefore they seem to be an interesting object of scientific research.

In this dissertation thesis the problem of boundary characterization of quasiregular counterparts of finite Blaschke products is considered. The complete solution of this problem is presented in Chapter 4 in the form of Theorems 4.4 and 4.7, which are the main results of this thesis. These theorems generalize Krzyż's boundary characterization of quasiconformal mappings of the unit disk $\mathbb{D}$ onto itself. A precise formulation of these theorems and their proofs require some auxiliary facts, that are described in details in Chapters 2 and 3. The first one refers to the functional equation

$$e^{i\varphi(z)} = f(e^{iz}) \quad z \in D,$$

with unknown function $\varphi : D \to \mathbb{C}$, where D is a given horizontal stripe and $f : D' \to \mathbb{C}\setminus\{0\}$ is a complex continuous function in the set $D' := \{e^{iz} : z \in D\}$. The existence of global solutions of this equation under the additional injectivity assumption on the function f is presented in Section 2.2. In Section 2.3 the injectivity of these solutions is studied. The results presented in Sections 2.1–2.3 are used in Section 2.4 to define the orientation of homeomorphisms in $E(\mathbb{C})$. Moreover, in this section the geometric characterization and basic properties of quasiconformal mappings in $E(\mathbb{C})$ are reminded.

The main purpose of Chapter 3 is a generalization of the classical Fatou theorem for finite Blaschke products to their quasiregular counterparts. Theorem 3.12 is the fundamental result in this direction and very useful for the considerations in Chapter 4. This theorem is a variant of Vourinen's theorem and it is proved in Section 3.3. Two previous sections are crucial in the proof of Theorem 3.12. In Section 3.1 necessary properties on the degree of a continuous function at the point are provided. These are known facts. However, due to completeness of the considerations, they were precisely derived as an application of the results presented in Sections 2.1–2.3. Section 3.2 contains a variant of the Fatou theorem in which the boundary condition was weakened and the information about degrees of zeros was included.

In Chapter 4 there are presented the main results of the PhD thesis. In Sections 4.1 and 4.2 detailed proofs of Theorems 4.4 and 4.7 are given, respectively. They generalize Theorems 4.1 and 4.2, proved by Krzyż; cf. Remarks 4.5 and 4.8. The generalization is to replace quasiconformal mappings of the unit disk $\mathbb{D}$ onto itself by complex quasiregular mappings f in $\mathbb{D}$ such that $|f(0)| \neq 1$ and $\mathrm{CS}(f) \subset \mathbb{T}$, where $\mathrm{CS}(f)$ is the cluster set of a function f on the boundary $\mathbb{T}$. Section 4.3 provides complementary remarks to previous two sections. Two interesting examples of applications of Theorem 3.12 are indicated by Corollaries 4.12 and 4.13. First of them deals with the Schwarz inequality, while the second one is relevant to Morii's inequality for complex quasiregular mappings f in the unit disk $\mathbb{D}$, which satisfy the boundary condition $\mathrm{CS}(f) \subset \mathbb{T}$.

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