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The summary of the PhD thesis

Selected properties and applications of direct sums of Banach spaces

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The aim of the thesis is to present original results concerning selected properties of direct sums of Banach spaces and their applications.

One of the problems studied in the Banach space theory concerns conditions which guarantee that a given geometric property of Banach spaces is preserved under some constructions, such as direct sums or interpolation spaces. The main part of the thesis is focused on results in this direction.

In particular we consider geometric properties of general direct sums of Banach spaces of the form $Y = (\sum_{i \in I} X_i)_E$, where $\{X_i\}_{i \in I}$ is a family of Banach spaces and the norm in the direct sum comes from the Banach lattice E called a basic space for the direct sum. Such sums with E being a Banach space with an unconditional basis with the unconditional constant one is used in the general discrete interpolation K -method which is considered in the last chapter of the thesis.

The first chapter of the thesis contains definitions and basic theorems concerning geometric properties of Banach spaces: uniform convexity, uniform smoothness, uniform nonsquareness, Opial properties and the García-Falset property. These geometric properties have many applications in functional analysis, especially in the metric fixed point theory. In the thesis three cases of the Opial property are considered: the non-strict Opial property, Opial property and uniform Opial property. They are studied not only with respect to the weak topology but also more generally with respect to some abstract topology τ . In the thesis, a new modulus $s_{X,\tau}$ of a Banach space X , related to the Opial property is defined. Moreover, we establish relations between this modulus and another one, already known modulus $r_{X,\tau}$ that is also related to the Opial property.

The second chapter is devoted to Banach lattices. Such lattices form a special class of Banach spaces in which the norm satisfies conditions related to the order. The chapter

contains an introduction to the Banach lattice theory and description of selected issues of this theory. Furthermore, properties of Banach lattices connected to the order are considered. Among them we consider uniform monotonicity and order uniform smoothness that may be described by functions called moduli of monotonicity $\delta_{m,X}$ and σ_X and the modulus of order smoothness $\rho_{m,X}$. Relations between these moduli are given. Also, a construction of a two-dimensional Banach lattice with the non-convex modulus of monotonicity is presented. This construction disproves claims appearing in the literature.

In the third chapter of the thesis we present original results on geometric properties of direct sums of Banach spaces. In particular an estimate for the modulus of convexity δ_Y of the direct sum $Y = (\sum_{i \in I} X_i)_E$ in terms of moduli of convexity of the spaces X_i and the space E is given. Furthermore, an estimate for the characteristic of convexity $\varepsilon_0(Y)$ of the space Y is given. The chapter contains also a construction of the space Y with the given values of $\varepsilon_0(Y)$ and $\delta_Y(2)$. Next, we prove results on the Opial properties for direct sums. They give sufficient conditions for a direct sum to have the non-strict Opial property, Opial property and uniform Opial property. The chapter ends with the original results related to the García-Falset coefficient for a direct sum.

The last chapter of the thesis is devoted to uniform convexity and the Opial property for interpolation spaces whose construction is based on the general discrete interpolation method. In this method direct sums with respect to a space with an unconditional basis is used. In our first result we give conditions for an interpolation space to be uniformly convex. The second one shows which conditions guarantee that an interpolation space has the Opial property and uniform Opial property.

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