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The Strong Law of Large Numbers for sums of random variables and random fields

Phd thesis in the field: Probability Theory,

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Summary

The Strong Law of Large Numbers (SLLN) is one of the most fundamental and widely used theorems of probability theory. SLLN are used, among others, in the following problems: calculating integrals with Monte Carlo methods (Stanisław Ulam used this method in calculation related to nuclear bomb), computations of empirical distribution function, proving theorems in number theory, constructing estimators in statistics, studying ergodic processes in physics or economics, such as, for example, the Ising model. SLLN lived to see many generalizations. Among other things, there were considered multidimensional sums of fields of random variables ([11], [12]), sums of arbitrarily dependent random variables but with the same law ([13]), randomly indexed sums ([9]) or were investigated the speed of convergence in SLLN (Baum-Katz type theorems, [14]).

Let V be an area in $\mathbb{N}^d$ such that the elements of main diagonal $(\underbrace{n, n, \dots n}_{d-times})$, $n \in \mathbb{N}$ belong to V. The SLLN for indices in a restricted domain relies on investigation

$$\lim_{\substack{|\underline{n}|\to\infty \\ \underline{n}\in V}} \frac{S_{\underline{n}}}{|\underline{n}|},$$

where $\underline{n} = (n_1, n_2, \dots n_d) \in \mathbb{N}^d$, $|\underline{n}| = \prod_{j=1}^d n_j$, $\{X_{\underline{n}}, \underline{n} \in \mathbb{N}^d\}$ is a field of independent identically distributed random variables and $S_{\underline{n}} = \sum_{\underline{k}\leqslant\underline{n}} X_{\underline{k}}$. In this case some technical complications in the proof arise (for details cf. in chapter 9.4 pp. 246-247 [11]). The proving techniques are to represent each term $X_{\underline{n}}$ as a finite linear combination of sums $S_{\underline{k}}, \underline{k} \leqslant \underline{n}$, and then to apply the SLLN for each sum, whence we get the almost sure convergence of $X_{\underline{n}}/|\underline{n}|$. Then we use the Borel-Cantelli Lemma for independent random events. However, in this linear combination, in the case of an irregular edge of V, it can be arised a lot of sums whose indices do not belong to V ($S_{\underline{k}}, \underline{k} \leqslant \underline{n}, \underline{k} \notin V$). In the first chapter of the work we give a new (has not found in the literature) partial solution of this problem, applying smoothing up and down the boundary of V and evaluating the difference between this two limits.

Randomly indexed sums, or sums for non-random subsequences, were considered in the literature, but SLLN for sums of randomly selected components has not been considered so far. Let $\{A_n, n \geqslant 1\}$ be a sequence of arbitrary dependent, almost sure finited, random subsets of $\mathbb{N}$, let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables independent of $\{A_n, n \geqslant 1\}$. SLLN for sums of randomly selected terms means investigation of a limits $\{\frac{S(A_n)}{b_n}, \frac{V(A_n)}{b_n}, \frac{Z(A_n)}{b_n}, n \geqslant 1\}$, where

$$\begin{aligned} S(A) &= \sum_{i \in A} X_i - E \sum_{i \in A} X_i, \\ V(A) &= \sum_{i \in A} X_i - \sum_{i \in A} EX_i, \\ Z(A) &= \sum_{i \in A} EX_i - E \sum_{i \in A} X_i. \end{aligned}$$

The results of this type are obtained in second chapter of the thesis, using techniques based on the Hájek-Rényi inequality described in [3].

In the third chapter of the thesis we prove SLLN for the random field $\{X_{\underline{n}}, \underline{n} \in \mathbb{N}^d\}$ of dependent random variables but with the same law. The summation is over the field of random sets $\{A_{\underline{n}}, \underline{n} \in \mathbb{N}^d\}$, however, in this chapter, the random sets are measurable mappings of a probabilistic space $(\Omega, \mathcal{A}, \mathcal{P})$ to $(\mathbb{N}^d, 2^{\mathbb{N}^d})$ and, as previous, almost sure finited. The obtained results generalize the one of Rosalsky and Stoica [13], who considered this problem for a sequence of random variables and non-random sums.

One of a types of SLLN is the Almost Sure Central Limit Theorem (ASCLT), where we sum up properly normalized indicators of dependent events. The history of this problem can be found, for example, in the monograph [10], while main technics of a proof are described in [1] and [2]. In the fourth chapter, as in the previous one, we consider sums over random subsets of $\mathbb{N}^d, d \geqslant 1$, more precisely, we examine sums $D_{\underline{n}}^{-1} \sum_{\underline{k} \leqslant \underline{n}} d_{\underline{k}} I[\frac{S(A_{\underline{k}})}{b_{\underline{k}}} < x]$ for some fields of positive reals $\{d_{\underline{n}}, b_{\underline{n}}, \underline{n} \in \mathbb{N}^d\}, D_{\underline{n}} = \sum_{\underline{k} \leqslant \underline{n}} d_{\underline{k}}$.

The last chapter of the work is devoted to the speed of convergence in the SLLN. We obtain the Baum-Katz theorem for a random (was not considered yet) d dimensional moving average process. This result generalize the main result of Sung [14], who considered the case of non-random moving average process in a one-dimensional space.

The results of the first chapter are published in [5], the ones of the second chapter are accepted to print in [8], the third chapters ones are reviewing in the journal *Periodica Mathematica Hungarica* [7] and partially are accepted for printing in [8], the results of the chapter fourth have been accepted for print in [6] and the results of the fifth chapter have appeared in [4].

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