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Abstract

Conservation laws in the modelling of collective phenomena

Over the last decades, traffic congestion, car accidents and pollution became daily issues. To understand and overcome road traffic problems, scientists from different research fields are creating advanced mathematical models. Mathematical models help to understand road traffic phenomena, develop optimal road network with efficient movement of traffic and minimal traffic congestion. This thesis is devoted to macroscopic traffic flow modelling, which describes traffic flow by variables averaged over multiple vehicles: density, velocity and flow. Macroscopic models naturally lead to conservation laws, which are hyperbolic partial differential equations. In recent years, this class of equations is more widely considered, but few theoretical results are available. This is caused by two main difficulties. The former is the non-linear hyperbolic nature of equations, which leads to consider weak solutions, instabilities and diffusivity of numerical schemes. The latter is the non-uniqueness of weak solutions and the need to introduce exotic functionals to select a unique physically reasonable solution.

In the first chapter, we introduce basic ideas of traffic modelling. First, we present the main classification of mathematical models with special attention to the level of details. Then we list the differences between the dynamics of traffic flow and that of flowing particles. Next, we show the minimal requirements to construct a physically reasonable macroscopic traffic flow model. We define three macroscopic variables to describe traffic flow, namely (average)

density ρ, (average) speed v and (average) flow f. We derive the basic relation between them and formulate scalar conservation law. The chapter ends with a short presentation of the models under consideration, followed by the results obtained during my doctoral studies.

The second chapter is devoted to a detailed discussion of basic macroscopic traffic flow models. The first presented model is the model proposed by Lighthill, Witham [7] and Richards [10] (LWR). It describes the dynamics of traffic via a scalar conservation law under the hypothesis that $v = v(\rho)$. We define a rarefaction wave, a shock wave and a contact discontinuity for the LWR model, and define the Riemann solver $\mathcal{RS}_{\mathrm{LWR}}$. In the end, we give a list of drawbacks of the LWR model.

The next considered model is the Aw, Rascle [1] and Zhang [11] model(ARZ). The ARZ model consists of two conservation laws, expressing the conservation of the number of vehicles and the conservation of the generalized momentum. We give the basic properties of the system, such as eigenvalues, eigenvectors and the corresponding Lagrangian markers. Next, we construct the Riemann solver $\mathcal{RS}_{\mathrm{ARZ}}$ using elementary waves. Finally, we give definitions of weak and entropy solutions for the ARZ model corresponding to $\mathcal{RS}_{\mathrm{ARZ}}$.

In the last part of this chapter, we describe models with phase transition (PT). The PT model treats differently traffic with low and high densities, on the basis of empirical studies. For this reason, we consider PT model described by the LWR model on the set Ω_{f} corresponding to the low densities and a 2×2 system of conservation laws on the set Ω_{c} corresponding to the high densities. We present two PT models, denoted by PT^a and PT^p, and introduced in [3, 6]. Then we recall from our paper [5] the generalization of these models for cases with a metastable phase ($\Omega_{\mathrm{f}} \cap \Omega_{\mathrm{c}} \neq \emptyset$) and without a metastable phase ($\Omega_{\mathrm{f}} \cap \Omega_{\mathrm{c}} = \emptyset$). Next, we introduce a notion of admissible solution for the Riemann problems and then Riemann solvers $\mathcal{RS}_{\mathrm{R}}$ and $\mathcal{RS}_{\mathrm{S}}$ accordingly.

The chapter ends with propositions regarding consistency and $\mathbf{L}^1_{loc}$-continuity for the Riemann solvers $\mathcal{RS}_{\rm R}$ and $\mathcal{RS}_{\rm S}$.

In the third chapter, we describe the LWR model with a local point constraint on the flow. More precisely, we consider a situation in which the maximum flow of cars is limited at a fixed point along the road. Thanks to such considerations, we can model traffic flow through toll gates or construction sites. We define the Riemann solver $\mathcal{CRS}_{\rm LWR}$ and list its main properties. Then we define the entropy solution of the Cauchy problem and recall the corresponding existence result.

The fourth chapter is devoted ARZ model with local point constraint on the flow and our results obtained in [8]. In our work we prove the existence of the weak solutions, corresponding to a non-conservative Riemann solver, in the class of functions with bounded variation. The goal is obtained by showing the convergence of a sequence of approximate solutions constructed via the Wave Front Tracking method. More precisely, we introduce grid, approximate Riemann solver $\mathcal{CRS}^n_{\rm ARZ}$ by splitting a rarefaction wave and construct approximate Cauchy problems. Thanks to the decreasing in time functional Υ, we show that the total variation of the approximated solution is uniformly bounded. By Helly's theorem we obtain convergence of approximated solutions and then we show that the limit function is indeed a weak solution to the Cauchy problem for the ARZ model with local point constraint on the flow.

In the fifth chapter, we describe the models PT^a and PT^p with the local point constraint on the flow and present our results obtained in [5, 2]. More precisely, we introduce Riemann Solvers $\mathcal{CRS}_{\rm R}$ and $\mathcal{CRS}_{\rm S}$, both with a metastable phase and without a metastable phase. Then we examine their consistency, $\mathbf{L}^1_{loc}$-continuity and invariant domains. The remainder of the chapter is devoted to the existence result of a weak solution in the class of function with bounded variation for the PT^p model with a metastable phase. The goal is obtained by showing the convergence of a sequence of approximate solutions constructed via Wave Front Tracking method. Similarly to the results from the previous chapter, we define grid and approximate Riemann solver $\mathcal{CRS}_{\mathrm{R}}^{p,n}$. Then we introduce the decreasing in time function $\mathtt{T}$ and show that the approximate solution has bounded variation, the number of waves and interactions is finite in finite time. We apply Helly's theorem and then show that the limit function is an entropy solution of the Cauchy problem for the PT^a model with the metastable phase.

The sixth chapter is devoted to the results obtained in conference proceedings [9, 4]. We consider there two macroscopic models on road networks. The former is the LWR model with moving constraint on the flow. The concept of moving constraint on the flow allows us to model situations in which a truck (or other slower vehicle) reduces the flow at its position. From a mathematical point of view, the constraint is given by the ordinary differential equation depending on the trajectory of the truck. We give a detailed description of the model for a unidirectional road, introduce a Riemann solver $\mathcal{BRS}_{\mathrm{LWR}}$ and generalize it for the case of road networks. The latter considered model is the PT model introduced in the second chapter. We generalize it to the case of road networks by introducing an appropriate Riemann solver.

At last, for the sake of clarity and to ease of comprehension, we defer to the appendix technical proofs.

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