Szymon Dudek Rzeszów, 12.06.2018 r.

Summary PhD thesis

# "Fixed point theorems and measures of noncompactness in Banach and Fréchet spaces and their applications to nonlinear integral equations".

Author: mgr Szymon Dudek
Supervisor: dr hab. Leszek Olszowy, prof. PRz

The purpose of this PhD thesis is to establish new fixed point theorems for operators in the Banach spaces and Fréchet spaces and their applications for the study of existence and character of solutions of nonlinear integral equations. Moreover, we also pay attention to how our theorems generalize older results and what benefits we can get when we apply them to the integral equations.

In the first section some basic definitions, properties and facts are presented. We also introduce basic symbols used in the dissertation.

In the second section we present an idea of a measure of noncompactness and we recall examples of the classical measures of noncompactness (i.e. Kuratowski measure of noncompactness, Hausdorff measure of noncompactness) with their properties. Furthermore, we present an axiomatic approach to the concept of the measure of noncompactness in Banach spaces as well as in Fréchet spaces. In the last part of this section we provide some examples of measures of noncompactness and weak measures of noncompactness in certain function Banach spaces and Fréchet spaces with their properties. These measures allow us to prove some existence theorems for the integral equations and investigate some important properties of the solutions (i.e. asymptotic stability, ultimate monotonicity).

In the third section the so-called condition $(m)$ is recalled. We discuss this condition in a few function Banach algebras and Fréchet algebras. It turns out that the Kuratowski measure of noncompactness, defined on the Banach algebra, satisfies this condition. This result is proved in [2]. If we talk about algebra of continuous functions on the compact interval or on $\mathbb{R}_+$, we will consider the product of two vectors as the usual product of the two functions. Then the measures of noncompactness described in the second section will be used in condition $(m)$. Moreover, we consider this condition in the spaces of absolutely integrable functions and locally absolutely integrable functions on $\mathbb{R}_+$ with the weak measures of noncompactness. Then we use the product as a convolution.

The fourth section contains the fixed point theorems for operators defined in Banach or Fréchet algebras. The condition $(m)$ and fixed point theorems of Darbo type are used to prove these theorems. We provide some applications of these theorems to existence

theorems for selected nonlinear integral equations in certain Banach algebras or Fréchet algebras. In these equations there are Volterra, Volterra-Hammerstein and fractional integral operators. Additionally, in some cases we describe the characterization of these solutions with the use of the measures of noncompactness.

The next section is devoted to presenting new fixed point theorems of Darbo type in Fréchet spaces. We discuss how these theorems generalize the classical Darbo theorem. Namely, we can assume weaker conditions for the operator $T$. It is sufficient to assume that some closed and convex iterations of image of the operator $T$ satisfy Darbo condition. It is not necessary to assume that operator $T$ fulfills this condition. Moreover, we provide the example of integral equations which shows that if we apply these new theorems, some assumptions can be omitted in contrast to the situation when we use older (weaker) versions of Darbo theorem. The above considerations were published in the papers [2] and [5].

In the sixth section we prove some generalization of Sadovskii's fixed point theorem in Fréchet spaces. In the second part of this section the definitions of the $\mu-$quasi-condensing mapping and $\mu-$FPP are recalled. Additionally, we give the answer to some open problem connected with these terms.

In the last section we investigate continuous dependence of a set of solutions of some nonlinear quadratic Volterra integral equation on a parameter. We consider this equation in the space $C([0,T])$ as well as in the space $C(\mathbb{R}_+)$. Furthermore, it is worth mentioning that we have not assumed the uniqueness of the solution of this integral equation. This fact implies that Hausdorff pseudometric is needed in our investigations.

Most of the results presented in the submitted dissertation were published in the papers [1] - [5].

# Bibliografia


[1] J. Banaś, S. Dudek, *The technique of measures of noncompactness in Banach algebras and its applications to integral equations*, Abstr. Appl. Anal. 2013, Article ID 537897 (2013).

[2] S. Dudek, *Fixed point theorems in Fréchet algebras and Fréchet spaces and applications to nonlinear integral equations*, Appl. Anal. Discrete Math. 11 (2017) 340–357.

[3] S. Dudek, *Measures of Noncompactness in a Banach Algebra and Their Applications*, J. Math. Appl. 40 (2017), 69–84.

[4] S. Dudek, L. Olszowy, *Continuous dependence of the solutions of nonlinear integral quadratic Volterra equation on the parameter*, Journal of Function Spaces, 2015, Article ID 471235.

[5] S. Dudek, L. Olszowy, *On generalization of Darbo-Sadovskii's type fixed point theorems for iterated mappings in Fréchet spaces*, J. Fixed Point Theory Appl. (w recenzji).


Szymon Dudek