AUTOREFERAT

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Contents

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      My Ph.D. thesis has been announced as distinguished by the Council of the Institute resolution. An extended version of my Ph.D. thesis has been published by CRC Press in the form of a book entitled *Classification of Lipschitz Mappings* for which I was awarded (first prize) by the rector of Maria Curie-Skłodowska University in Lublin, Poland.
    - Postgraduate studies in the Institute of Mathematics at Maria Curie-Skłodowska University in Lublin, Poland, 2007–2011.
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      My Master's thesis won the award (second prize) of the Polish Mathematical Society in honour of Józef Marcinkiewicz (1910-1940) for the best student thesis on any branch of mathematics. I published the results contained in it in *Topological Methods in Nonlinear Analysis* and *Nonlinear Analysis*.
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My main scientific achievements (according to *art. 16 ust. 2 ustawy z dnia 14 marca 2003 r. o stopniach i tytule naukowym oraz stopniach i tytule w zakresie sztuki*) are a series of six monothematic publications entitled

Recent developments in the theory of Lindenstrauss spaces

[CMP2015] E. Casini, E. Miglierina, and Ł. Piasecki, *Hyperplanes in the space of convergent sequences and preduals of ℓ_1*, Canad. Math. Bull. 58 (2015) 459-470.

[CMP2017] E. Casini, E. Miglierina, and Ł. Piasecki, *Separable Lindenstrauss spaces whose duals lack the weak* fixed point property for nonexpansive mappings*, Studia Math. 238 (1) (2017), 1-16.

[CMPV2016] E. Casini, E. Miglierina, Ł. Piasecki, and L. Veselý, *Rethinking polyhedrality for Lindenstrauss spaces*, Israel J. Math. 216 (2016), 355-369.

[CMPP2018] E. Casini, E. Miglierina, Ł. Piasecki, and R. Popescu, *Weak* fixed point property in ℓ_1 and polyhedrality in Lindenstrauss spaces*, Studia Math. 241 (2) (2018), 159-172.

[CMPP2017] E. Casini, E. Miglierina, Ł. Piasecki, and R. Popescu, *Stability constants for the weak* fixed point property in ℓ_1*, J. Math. Anal. Appl. 452 (1) (2017), 673-684.

[P2018] Ł. Piasecki, *On Banach space properties that are not invariant under the Banach-Mazur distance* 1, J. Math. Anal. Appl. 467 (2018), 1129-1147.

1. Introduction

It is a matter of general agreement that the $L_p(\mu)$ spaces ($1 \le p \le \infty$ and μ a measure) and the $C(K)$ spaces (K compact Hausdorff) are among the most important Banach spaces. A central part of Banach space theory is devoted to the investigation of the special properties of these spaces and some closely related spaces. This part of Banach space theory is often called the theory of the classical Banach spaces. It is our feeling that in order to get a well rounded theory of the classical Banach spaces, in the framework of the isometric theory, it is worthwhile to take as the main objects of the investigation the class of Banach spaces X for which $X^* = L_p(\mu)$ for some $1 \le p \le \infty$ and some measure μ. Let us examine briefly the relation of this latter class of spaces to those mentioned in the first sentence. Since for $1 < p < \infty$ the $L_p(\mu)$ spaces are reflexive it is clear that $X^* = L_p(\mu)$ if and only if $X = L_q(\mu)$ ($p^{-1} + q^{-1} = 1$). Grothendieck proved the non obvious fact that if $X^* = L_\infty(\mu)$ then $X = L_1(\mu)$. Well-known results of F. Riesz and Kakutani show that if $X = C(K)$ then $X^* = L_1(\mu)$ for a suitable μ. There are, however, Banach spaces X which are not isometric to $C(K)$ spaces while their duals are $L_1(\mu)$ spaces. These are thus the only spaces which should be included in the geometric theory of the classical Banach spaces and which are not "classical" in the strict sense.

A. J. Lazar, J. Lindenstrauss, *Banach spaces whose duals are L_1-spaces and their representing matrices*, Acta Math. 126 (1971), 165-194.

A real Banach space X for which $X^* = L_1(\mu)$ for some measure μ is named an *L_1-predual* or a *Lindenstrauss space*. In particular, if $X^* = \ell_1$, then it is called an *ℓ_1-predual*.

In the last four years, the main subject of my mathematical interest was studying the geometrical and topological properties of separable Lindenstrauss spaces. Intense collaboration with my scientific partners from Italy, Emanuele Casini from Università dell'Insubria in Como and Enrico Miglierina from Università Cattolica del Sacro Cuore in Milan, and later also with Libor Veselý from Università di Milano, and Roxana Popescu from University of Pittsburgh (PA, USA), resulted in a series of new and interesting results. In particular, considerations on some classical issues of *metric fixed point theory* led us to a surprising discovery in a different, till now separately studied *theory of polyhedral spaces*. Namely, it turned out that some classical characterizations of polyhedral Lindenstrauss spaces are false, whereas some others have incorrect proofs. We took care of ordering, rebuilding and developing this theory, focusing primarily on the indication of geometric equivalences of polyhedral properties for ℓ_1-preduals.

The results obtained by us form now a complete theory, and its subsequent stages of rising I will discuss in the autoreferat.

2. Hyperplanes in c and preduals of ℓ_1

Let X be a real Banach space. By B_X and S_X we denote the closed unit ball and the unit sphere in X, respectively. X^* stands for the dual space of X. If $A \subset X$, then $\operatorname{conv}(A)$, $\operatorname{ext}(A)$, $\operatorname{int}(A)$, $\overline{A}$, $[A]$ and $A^\perp$ denote the convex hull of A, the set of all extreme points of A, the interior of A, the closure of A in X, the closed linear span of A in X and the annihilator of A in X^*, respectively. If $A \subset X^*$, then by $\overline{A}^*$ we denote the weak* closure of A, and by A' the set of all weak* cluster points of A:

$$A' = \left\{ x^* \in X^* : x^* \in \overline{(A \setminus \{x^*\})}^* \right\}.$$

We say that a linear subspace Y of X is 1-*complemented in* X if there exists a linear projection P from X onto Y with $\|P\| = 1$. If $f \in X^*$, then $\ker f$ stands for the kernel of functional f, that is, $\ker f = \{x \in X : f(x) = 0\}$. If X contains an isometric copy of Y, then we write $Y \subset X$. Whenever X is isometrically isomorphic to Y, we write $X = Y$.

E. Michael and A. Pełczyński [64] and A. J. Lazar and J. Lindenstrauss [51] proved that a separable Banach space X satisfies $X^* = L_1(\mu)$ for some measure μ if and only if X has a monotone basis $\{x_i\}_{i=1}^\infty$ such that for every $n \in \mathbb{N}$ the subspace $[\{x_i\}_{i=1}^n]$ is isometric to ℓ_∞^n (i.e. the space $\mathbb{R}^n$ with the norm $\|x\| = \|(x(1), x(2), \ldots, x(n))\| = \max_{1 \le i \le n} |x(i)|$). Although this result fully characterizes Lindenstrauss spaces for the class of separable Banach spaces, in many situations it is more convenient to use a concrete model of such a space.

The space c_0 with the standard maximum norm is the classical example of ℓ_1-predual. Here, the duality mapping $\phi : \ell_1 \to c_0^*$ is given by

$$(\phi(y))(x) = \sum_{j=1}^{\infty} x(j) y(j)$$

for $y = (y(1), y(2), \ldots) \in \ell_1$ and $x = (x(1), x(2), \ldots) \in c_0$.

We will begin our considerations with a summary of the known results on ℓ_1-preduals hyperplanes in the space c_0:

Theorem 2.1 ([CMP2015], Theorem 1.1)**.** *Let $f \in \ell_1 = c_0^*$ be such that $\|f\|_{\ell_1} = 1$. Consider a hyperplane $V_f = \ker f \subset c_0$. The following properties are equivalent:*

(1) V_f is 1-complemented,
(2) V_f^* is isometric to ℓ_1,
(3) there exists j_0 such that $|f(j_0)| \ge \frac{1}{2}$,
(4) V_f is isometric to c_0.

The equivalence (1) $\Leftrightarrow$ (3) was proved by J. Blatter and E. W. Cheney [11]. The implication (1) $\Rightarrow$ (4) follows from the fact that 1-complemented infinite-dimensional subspaces of c_0 are isometric to c_0 (see, for example, [59]). Clearly, (4) $\Rightarrow$ (2). We show now the proof of the implication (2) $\Rightarrow$ (1). Observe first that V_f^* is isometric to the quotient space $\ell_1/[f]$, and so $V_f^{**} = [f]^\perp = \{x^{**} \in \ell_\infty : x^{**}(f) = 0\}$. The space V_f^{**} is isometric to ℓ_∞ because V_f^* is isometric to ℓ_1. Therefore, V_f^{**} is 1-complemented in ℓ_∞ ([23], Proposition 5.13). It is enough now to apply the result of M. Baronti ([6], Corollary 2) stating that V_f^{**} is 1-complemented in ℓ_∞ if and only if V_f is 1-complemented in c_0.

The space c of converging sequences with the supremum norm is another commonly known example of an ℓ_1-predual. In this case the duality mapping $\phi : \ell_1 \to c^*$ is defined by

$$(\phi(y))(x) = \sum_{j=0}^{\infty} x(j)y(j+1)$$

for $y = (y(1), y(2), \dots) \in \ell_1$, $x = (x(1), x(2), \dots) \in c$ and $x(0) = \lim_{j\to\infty} x(j)$.

We have already seen that all ℓ_1-preduals hyperplanes in c_0 are actually isometric to c_0. As we will see in a moment, the structure of ℓ_1-predual hyperplanes in c is much richer. We begin with the following result which can be seen as a counterpart of Theorem 2.1:

Theorem 2.2 ([CMP2015], Theorem 1.2)**.** *Let $f \in \ell_1$ be such that $\|f\|_{\ell_1} = 1$ and let $W_f = \ker f \subset c$. Consider the following properties:*

(1) W_f is 1-complemented;
(2) W_f is isometric to c;
(3) there exists $j_0 \geq 2$ such that $|f(j_0)| \geq \frac{1}{2}$;
(4) W_f^* is isometric to ℓ_1;
(5) there exists $j_0 \geq 1$ such that $|f(j_0)| \geq \frac{1}{2}$;
(6) W_f is isometric to c_0;
(7) $\inf_P \|P\| = 2$ (where $P : c \to W_f$ is a linear projection of c onto W_f);
(8) $|f(1)| = 1$, $f(j) = 0$ for every $j \geq 2$.

Then the following implications hold:

$$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Rightarrow (4) \Leftrightarrow (5) \Leftarrow (6) \Leftrightarrow (7) \Leftrightarrow (8).$$

The above theorem follows from the series of technical results proved in Sections 2 and 3, as well as in Section 4 (Proposition 4.1) in [CMP2015].

As a consequence, the set of all hyperplanes W_f in c satisfying $W_f^* = \ell_1$ can be divided into three distinct classes:

- W_f is isometric to c (equivalently, there exists $j_0 \geq 2$ such that $|f(j_0)| \geq \frac{1}{2}$);
- W_f is isometric to c_0 (equivalently, $|f(1)| = 1$);
- W_f is isometric neither to c nor c_0 (equivalently, $\frac{1}{2} \leq |f(1)| < 1$ and $|f(j)| < \frac{1}{2}$ for every $j \geq 2$).

It turns out that the most interesting situation is when W_f is isometric neither to c_0 nor c. For this class of spaces we study the behavior of the $\sigma(\ell_1, W_f)$-cluster points of the standard basis in ℓ_1 and we give an explicit formula for the isometrical isomorphism ϕ from ℓ_1 onto W_f^*.

Theorem 2.3 ([CMP2015], Theorem 4.3)**.** *Let $W_f \subset c$ be such that $\frac{1}{2} \leq |f(1)| < 1$ and $|f(j)| < \frac{1}{2}$ for every $j \geq 2$. Then the mapping $\phi : \ell_1 \to W_f^*$ given by*

$$(\phi(y))(x) = \sum_{j=1}^{+\infty} x(j)y(j)$$

for $y = (y(1), y(2), \dots) \in \ell_1$ and $x = (x(1), x(2), \dots) \in W_f$ is an isometrically isomorphism from ℓ_1 onto W_f^*. Moreover, if (e_n^*) denotes the standard basis in ℓ_1, then

$$e_n^* \xrightarrow{\sigma(\ell_1, W_f)} e^*,$$

where $e^* = \left(-\frac{f(2)}{f(1)}, -\frac{f(3)}{f(1)}, -\frac{f(4)}{f(1)}, \dots\right)$.

The above theorem has a very interesting consequence. Namely, for every point $e^* = (e^*(1), e^*(2), \dots) \in B_{\ell_1}$ we can choose a predual X of ℓ_1 such that

$$e_n^* \overset{\sigma(\ell_1,X)}{\longrightarrow} e^*.$$

Indeed, it is enough to consider $X = W_f$ with $f = (f(1), f(2), ...) \in \ell_1$ satisfying the following condition:

$$f(1) = \frac{1}{1 + \sum_{n=1}^{\infty} |e^*(n)|}, \quad f(n) = -\frac{e^*(n-1)}{1 + \sum_{n=1}^{\infty} |e^*(n)|} \quad \text{for every } n \geq 2.$$

Moreover, the above choice of a predual space is unequivocal, that is, if X is an ℓ_1-predual such that $e_n^* \overset{\sigma(\ell_1,X)}{\longrightarrow} e^*$, then X must be isometric to W_f (see Corollary 4.4 in [CMP2015]).

Reassuming, all ℓ_1-preduals for which the standard basis in ℓ_1 is weak*-convergent are located among the hyperplanes in the space c.

The last part of the paper [CMP2015] refers to the result of W. B. Johnson and M. Zippin [41] stating that every separable L_1-predual is isometric to a quotient of $C(\triangle)$, where $\triangle$ denotes the Cantor set. The question whether for any ℓ_1-predual X there exists a countable and compact metric space K such that X is isometric to a quotient of $C(K)$ was settled by Alspach [3]. Namely, he gave an example of an ℓ_1-predual hyperplane in the space c, which is not isometric to a quotient of any space $C(\alpha)$, where α denotes a countable ordinal number and by $C(\alpha)$ we mean, as usual, the space of all continuous functions on the set of ordinals less than or equal to α with the order topology. Let's recall here the classical Mazurkiewicz and Sierpiński's result [63] which says that every space $C(K)$, where K is a countable and compact metric space, is isometric to $C(\alpha)$ for some α.

The following result, which can be seen as an extension of Alspach's example, characterizes all ℓ_1-preduals hyperplanes in c having this property.

Corollary 2.4 ([CMP2015], Corollary 4.5)**.** *There exists a countable ordinal number α such that W_f is isometric with a quotient of $C(\alpha)$ if and only if one of the following two conditions hold:*

(1) there exists $j_0 \geq 2$ such that $|f(j_0)| \geq \frac{1}{2}$;
(2) $\frac{1}{2} \leq |f(1)| \leq 1$ and $|f(j)| < \frac{1}{2}$ for all $j \geq 2$ and $f = (f(1), f(2), \dots, f(n), 0, 0, \dots)$ for some $n \in \mathbb{N}$.

3. The weak* fixed point property in ℓ_1

Let X be an infinite-dimensional Banach space. We say that a nonempty bounded closed and convex subset C of X has the *fixed point property* (briefly, FPP) if every *nonexpansive* mapping $T : C \to C$ (i.e. $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$) has a fixed point. The space X has the *fixed point property* (briefly, FPP) if every nonempty bounded closed and convex subset C of X has the FPP. The space X has the *weak fixed point property* (briefly, w-FPP) if every nonempty, weakly compact, convex set $C \subset X$ has the FPP. The dual space X^* has the *weak* fixed point property* (briefly, w^*-FPP or $\sigma(X^*, X)$-FPP) if every nonempty, convex, $\sigma(X^*, X)$-compact set $C \subset X^*$ has the FPP.

Clearly, for every Banach space X the following implication holds:

$$X \text{ has the FPP} \Rightarrow X \text{ has the } w\text{-FPP}.$$

Moreover, for the dual space X^* we have:

$$X^* \text{ has the FPP} \Rightarrow X^* \text{ has the } w^*\text{-FPP} \Rightarrow X^* \text{ has the } w\text{-FPP}.$$

If X is reflexive, then the above three properties are equivalent.

Metric Fixed Point Theory is widely studied in the book of K. Goebel and W. A. Kirk [31] and Handbook [44]. Here, we will only cite those of the known results that concern the classical Banach spaces.

- $L_p(\mu)$ has the FPP for $p \in (1, \infty)$ (F. E. Browder, [17], [18], D. Göhde [36], W. A. Kirk [43]).
- $L_1(0,1)$ fails the w-FPP (D. E. Alspach, [2]). Consequently, the spaces $C[0,1]$, ℓ_∞ and $L_\infty(0,1)$ fail the w-FPP.
- L_1-preduals fail the FPP. This is due to Zippin's claim [78] that every such space contains an isometric copy of c_0 (the space c_0 fails the FPP; indeed, it is easy to observe that a mapping $T : B_{c_0} \to B_{c_0}$ defined by $T(x(1), x(2), \dots) = (1, x(1), x(2), \dots)$ is a fixed point free isometry).
- c_0 and c enjoy the w-FPP (B. Maurey, [61]).
- $C(\omega^n + 1)$ has the w-FPP, where $n \in \mathbb{N}$ and ω is the first infinite ordinal number (J. Elton, P. K. Lin, E. Odell, S. Szarek, [22]).
- If X is a separable Lindenstrauss space such that X^* is nonseparable, then X lacks the w-FPP. Indeed, Lazar and Lindenstrauss [52] proved that every such space X contains an isometric copy of $C(\Delta)$, where Δ is the Cantor set. Consequently, X contains an isometric copy of $L_1(0,1)$, and so, by Alspach's result, X fails the w-FPP.
- ℓ_1 has the w-FPP; moreover, by Schauder's fixed point theorem [73] and the fact that ℓ_1 satisfies Schur's property, every continuous self-mapping on a nonempty, weakly compact and convex set in ℓ_1 has a fixed point.
- ℓ_1 has the $\sigma(\ell_1, c_0)$-FPP (L. A. Karlovitz, [42]).
- ℓ_1 lacks the $\sigma(\ell_1, c)$-FPP. Indeed, the set
$$S^+ = \left\{ (x(1), x(2), \dots) \in \ell_1 : \sum_{i=1}^{\infty} x(i) = 1, x(i) \geq 0, i = 1, 2, \dots \right\}$$
is convex, $\sigma(\ell_1, c)$-compact and the mapping $T : S^+ \to S^+$ given by
$$T(x(1), x(2), \dots) = (0, x(1), x(2), \dots)$$
is a fixed point free isometry.
- $C(K)^*$ fails the w^*-FPP, when K is an infinite, compact Hausdorff space (M. Smyth, [75]).
- Let τ be a locally convex topology in the space ℓ_1 coarser than the weak topology on the unit ball. Assume that the standard basis (e_n) converges to some $e \in \ell_1$ with respect to τ. Then ℓ_1 has the τ-FPP if and only if one of the following conditions holds:
  (1) $\|e\| < 1$;
  (2) $\|e\| = 1$ and the set $N^+ = \{n \in \mathbb{N} : e(n) \geq 0\}$ is finite.
  The above result was proved by M. A. Japón-Pineda and S. Prus ([39], Theorem 8) and it includes those among the weak* topologies in ℓ_1 for which the standard basis (e_n^*) is weak* convergent. Recall that all ℓ_1-preduals having this property have been characterized in the previous chapter. As a result, we get the following:

Proposition 3.1 ([CMP2017], Proposition 2.2)**.** *Let $f \in \ell_1 = c^*$ be such that $\|f\| = 1$, $\frac{1}{2} \leq |f(1)| \leq 1$ and $|f(j)| \leq \frac{1}{2}$ for all $j \geq 2$. The space ℓ_1 has the $\sigma(\ell_1, W_f)$-FPP if and only if one of the following conditions holds:*

(1) $|f(1)| > \frac{1}{2}$,
(2) $|f(1)| = \frac{1}{2}$ and the set $N^+ = \{n \in \mathbb{N} : f(1)f(n+1) \leq 0\}$ is finite.

Remark 1 ([CMP2017], Remark 2.6)**.** In the case of a particular family of sets in the space ℓ_1, a characterization of those for which the FPP holds was given by K. Goebel and T. Kuczumow [32]. Consider the positive part of the unit sphere in ℓ_1:

$$S^+ = \overline{\text{conv}}\,\{e_i^* : i \in \mathbb{N}\} = \left\{ \sum_{i=1}^{\infty} \alpha_i e_i^* : \alpha_i \geq 0, i = 1, 2, \dots, \sum_{i=1}^{\infty} \alpha_i = 1 \right\}.$$

Let (a_i) be a bounded sequence of nonnegative numbers. Put $a = \inf a_i$ and $N_0 = \{i : a_i = a\}$. Modify now the set S^+ by moving its vertexes along half-lines emanating from the origin:

$$C = \overline{\text{conv}}\,\{(1 + a_i)e_i^* : i \in \mathbb{N}\} = \left\{ \sum_{i=1}^{\infty} \alpha_i (1 + a_i) e_i^* : \alpha_i \geq 0, i = 1, 2, \dots, \sum_{i=1}^{\infty} \alpha_i = 1 \right\}.$$

Sets of this form are nowadays called *Goebel-Kuczumow sets*. In [32] the authors proved that a set C has the FPP if and only if N_0 is nonempty and finite.

It turns out that many sets of this type are weak* compact with respect to appropriately chosen weak* topology that have the FPP. To illustrate this, for every $\varepsilon \in (0, 1)$ define the set

$$C_\varepsilon = \left\{ \alpha_1 (1 - \varepsilon) e_1^* + \sum_{i=2}^{\infty} \alpha_i e_i^* : \alpha_i \geq 0, \sum_{i=1}^{\infty} \alpha_i = 1 \right\}.$$

The set C_ε is convex, bounded and closed. Moreover, it has the FPP because $\frac{1}{1-\varepsilon} C_\varepsilon$ has the FPP. Obviously, C_ε is neither $\sigma(\ell_1, c)$-compact nor $\sigma(\ell_1, c_0)$-compact. Nevertheless, for $f = \left(\frac{1}{2-\varepsilon}, -\frac{1-\varepsilon}{2-\varepsilon}, 0, 0, \dots\right)$, $W_f^* = \ell_1$ and

$$e_n^* \xrightarrow{\sigma(\ell_1, W_f)} (1 - \varepsilon) e_1^*.$$

Therefore C_ε is $\sigma(\ell_1, W_f)$-compact. By Proposition 3.1, ℓ_1 has the $\sigma(\ell_1, W_f)$-FPP.

The main purpose of this chapter is to characterize all separable Lindenstrauss spaces X with X^* failing the $\sigma(X^*, X)$-FPP. We will begin our consideration by analyzing the role played here by the space c.

Theorem 3.2 ([CMP2017], Theorem 3.2)**.** *If a separable Banach space X contains an isometric copy of c, then X^* fails the $\sigma(X^*, X)$-FPP.*

The above theorem extends aforementioned result of Smyth, and its proof is based on the following observation:

Proposition 3.3 ([CMP2017], Proposition 3.1)**.** *If a separable Banach space X contains an isometric copy of c, then X contains a subspace Y isometric to c and 1-complemented in X.*

Indeed, if $c \subset X$, then by the above proposition there exists a subspace Y isometric to c and a linear projection $P : X \to Y$ with $\|P\| = 1$. Then the adjoint operator $P^* : Y^* = c^* \to X^*$ is a weak* continuous isometry from c^* into X^*. Therefore, $P^*(S^+)$ is a convex, $\sigma(X^*, X)$-compact set which lacks the FPP.

Remark 2 ([CMP2017], Remark 3.3)**.** C. Lennard ([74], see Examples 3.2-3.3, p. 41-43) gave an example of convex, weak* compact set $C \subset c^*$ and a fixed point free *contractive* mapping $T : C \to C$ (i.e. $\|T(x) - T(y)\| < \|x - y\|$ for all $x, y \in C, x \neq y$). Therefore, under the assumptions of the previous theorem, X^* fails the $\sigma(X^*, X)$-FPP for contractive mappings.

Remark 3 ([CMP2017], Remark 3.5)**.** Let X be a separable Banach space. Suppose that there exists a quotient X/Y isometric to c. Theorem 3.2 shows that $Y^{\perp}$ fails the $\sigma(Y^{\perp}, X/Y)$-FPP and it follows easily that also X^* fails the $\sigma(X^*, X)$-FPP.

Observe that considering a quotient is a significant weakening of the assumption in Theorem 3.2. Indeed, in view of Proposition 3.3 every separable Banach space containing an isometric copy of the space c has a quotient isometric to c. Moreover, the space $X = \ell_1$ does not contain any copy of c but it has a quotient isometric to c.

Let us now return to the case of L_1-preduals. As we have already mentioned, Lazar and Lindenstrauss [52] proved that every separable L_1-predual whose dual is nonseparable contains a subspace isometric to $\mathcal{C}(\Delta)$, the space of continuous functions on the Cantor set Δ and equipped with the standard sup norm. Since $\mathcal{C}(\Delta)$ is a universal space for separable Banach spaces, it follows that it contains an isometric copy of c. We therefore obtain the following:

Corollary 3.4 ([CMP2017], Corollary 3.4)**.** *Let X be a separable Lindenstrauss space such that X^* is nonseparable. Then X^* lacks the $\sigma(X^*, X)$-FPP.*

It is known that if X is a Lindenstrauss space such that X^* is separable, then $X^* = \ell_1$. Therefore, the only case of interest to us is the class of ℓ_1-preduals, and in the context of the above-discussed results the following questions seem to be natural:

Suppose that X is an ℓ_1-predual such that ℓ_1 fails the $\sigma(\ell_1, X)$-FPP. Does X contain an isometric copy of the space c? Does X have a quotient that contains an isometric copy of the space c?

It turns out that the answer to the above questions is negative. The key result in this matter is the following

Proposition 3.5 ([CMP2017], Proposition 2.1)**.** *Let $f \in \ell_1 = c^*$ be such that $\|f\| = 1$ and $|f(1)| \geq \frac{1}{2}$. Then the following statements are equivalent:*

(1) W_f contains a subspace isometric to c.

(2) $|f(1)| = \frac{1}{2}$, the set $\{n \in \mathbb{N} : f(1)f(n+1) > 0\}$ is finite, and $\{n \in \mathbb{N} : f(n+1) = 0\}$ is infinite.

Example 1 ([CMP2017], Example 2.4)**.** Consider the hyperplane W_f, where

$$f = \left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \dots\right) \in \ell_1.$$

Then

- $W_f^* = \ell_1$ (see Theorem 2.3);
- W_f does not contain an isometric copy of c (see Proposition 3.5);
- ℓ_1 lacks the $\sigma(\ell_1, W_f)$-FPP (see Proposition 3.1).

This hyperplane has yet another feature that is important in the context of the main result of this chapter (Theorem 3.7). Namely,

- W_f does not have a quotient that contains an isometric copy of c.

The following example shows that to consider a quotient of X is a true extension of Theorem 3.2, even in the restricted framework of ℓ_1-preduals.

Example 2 ([CMP2017], Example 3.6)**.** Consider the space W_f where

$$f = \left(-\frac{1}{2}, \frac{1}{4}, 0, -\frac{1}{8}, 0, \frac{1}{16}, 0, \dots\right) \in \ell_1.$$

Then:

- $W_f^* = \ell_1$ (see Theorem 2.3);
- W_f does not contain an isometric copy of c (see Proposition 3.5).

Nevertheless, there exists a quotient of W_f isometric to c. Indeed, consider the subspace

$$Y = \{y \in W_f : y(2k) = 0 \text{ for all } k \in \mathbb{N}\}$$

and the map $T : c \longrightarrow W_f/Y$ defined by

$$T(x) = \left(\frac{7}{3}x(0), x(1), x(0), x(2), x(0), \dots\right) + Y$$

for every $x \in c$. One can show that T is an isometrically isomorphism.

The above examples suggest that the solution of our main problem is one of a very delicate nature. Because the spaces W_f whose duals fail the w^*-FPP will play an important role in our considerations, we introduce the following:

Definition 1 ([CMP2017], Definition 2.3)**.** We say that a space W_f is *bad with respect to w^*-FPP"* (briefly *bad*) if $f \in \ell_1$ is such that $\|f\| = 1$, $|f(1)| = \frac{1}{2}$ and the set $N^+ = \{n \in \mathbb{N} : f(1)f(n+1) \leq 0\}$ is infinite.

The next result is a more subtle version of Theorem 3.2:

Theorem 3.6 ([CMP2017], Theorem 3.7)**.** *Let X be a separable Banach space. If X contains an isometric copy of bad W_f, then X^* lacks the $\sigma(X^*, X)$-FPP.*

Remark 4 ([CMP2017], Remark 3.8)**.** Let X be a separable Banach space and suppose that a bad W_f is a subspace of a quotient X/Y of X. By Theorem 3.6 $Y^\perp$ fails the $\sigma(Y^\perp, X/Y)$-FPP. Consequently, X^* lacks the $\sigma(X^*, X)$-FPP.

We will now characterize the ℓ_1-preduals X such that ℓ_1 fails the $\sigma(\ell_1, X)$-FPP. This is the main result in [CMP2017]. The proof of implication (1) $\Rightarrow$ (4) is crucial here because it allows us to delete the restrictive assumption on the w^*-convergence of the standard basis of ℓ_1 used in the paper by M. A. Japón-Pineda and S. Prus ([39], Theorem 8).

Theorem 3.7 ([CMP2017], Theorem 4.1)**.** *Let X be a predual of ℓ_1. Then the following are equivalent:*

(1) ℓ_1 lacks the $\sigma(\ell_1, X)$-FPP for nonexpansive mappings.
(2) ℓ_1 lacks the $\sigma(\ell_1, X)$-FPP for isometries.
(3) ℓ_1 lacks the $\sigma(\ell_1, X)$-FPP for contractive mappings.
(4) There is a subsequence $(e^*_{n_k})_{k\in\mathbb{N}}$ of the standard basis $(e^*_n)_{n\in\mathbb{N}}$ in ℓ_1 which is $\sigma(\ell_1, X)$-convergent to a norm-one element $e^* \in \ell_1$ with $e^*(n_k) \geq 0$ for all $k \in \mathbb{N}$.
(5) There is a quotient of X isometric to a bad W_f.
(6) There is a quotient of X that contains a subspace isometric to a bad W_g.

Remark 5 ([CMP2017], Remark 4.2)**.** Recall that bad W_f and W_g in statements (5) and (6) of Theorem 3.7 cannot be replaced by c (see Example 1).

It is not known whether the lack of the $\sigma(\ell_1, X)$-FPP implies that X contains an isometric copy of a bad W_f.

4. Polyhedrality in Lindenstrauss spaces

Considerations regarding the weak* fixed point property, in particular Proposition 3.5, led us to a surprising discovery in the polyhedral theory.

Recall that a real Banach space X is called *polyhedral* if the closed unit ball of every finite-dimensional subspace of X is a polytope (i.e. it has a finite number of extreme points or, equivalently, it arises as the intersection of a finite number of closed half-spaces). This definition was introduced by Klee [47] who extended the notion of convex finite-dimensional polytope to the case of the closed unit ball B_X of an infinite-dimensional Banach space X. The space c_0 is a classical example of a polyhedral Lindenstrauss space (see [47], Proposition 4.7). Moreover, Lindenstrauss [57] proved that the dual of every infinite-dimensional Banach space is not polyhedral. In particular, every infinite-dimensional reflexive Banach space and the space ℓ_1 are not polyhedral. The space c is an example of another space which is not polyhedral. An elementary proof of this fact was given by Libor Veselý (see [38]).

We will give now an example of ℓ_1-predual whose closed unit ball has an extreme point but this space does not contain an isometric copy of c. This example disproves a result stated by Zippin in a paper [78] published in 1969. Let us remind the following

Fact 1. In Remark A of Section 4 of the paper [*M. Zippin, On some subspaces of Banach spaces whose duals are L_1 spaces, Proc. Amer. Math. Soc. 23 (1969), 378-385*] the author stated that every separable Lindenstrauss space with an extreme point contains an isometric (1-complemented) copy of c.

We will now give a necessary condition for the presence of an isometric copy of c in a separable Banach space:

Theorem 4.1 ([CMPV2016], Theorem 2.1)**.** *Let X be a separable Banach space. If X contains a subspace linearly isometric to c, then there exist $x \in X$ and a sequence $(v_n^*) \subset ext(B_{X^*})$ such that (v_n^*) is w^*-convergent to v^*, $v_n^*(x) = v^*(x) = \|v^*\| = \|x\| = 1$ and $\|v_n^* \pm v^*\| = 2$ for every $n \in \mathbb{N}$.*

Let us emphasize that the justification of Theorem 4.1 required using a completely different technique than the one in the proof of Proposition 3.5. If X is a predual of ℓ_1, we obtain the following

Corollary 4.2 ([CMPV2016], Corollary 2.2)**.** *Let X be a predual of ℓ_1. If X contains a subspace isometric to c then there exist $x \in B_X$ and a subsequence $(e_{n_k}^*)_{k\in\mathbb{N}}$ of the standard basis $(e_n^*)_{n\in\mathbb{N}}$ in ℓ_1 such that*

(1) $e_{n_k}^* \overset{\sigma(\ell_1,X)}{\longrightarrow} e^*$ *and* $supp\ e_{n_k}^* \cap supp\ e^* = \emptyset$ *for every* $k \in \mathbb{N}$, *where for* $x^* \in \ell_1 = X^*$ *we put* $supp\ x^* := \{i \in \mathbb{N} : x^*(i) \neq 0\}$,
(2) $e_{n_k}^*(x) = e^*(x) = 1$ *for every* $k \in \mathbb{N}$.

Example 3 ([CMPV2016], Section 3)**.** Consider the following hyperplane in c:

$$W = \left\{ x = (x(1), x(2), \dots) \in c : \lim_i x(i) = \sum_{i=1}^{\infty} \frac{x(i)}{2^i} \right\}.$$

By Theorem 2.3, $W^* = \ell_1$ and

$$e_n^* \overset{\sigma(\ell_1,W)}{\longrightarrow} e^* = \left(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots \right).$$

Therefore, by applying Corollary 4.2 (see also Proposition 3.5) we conclude that W does not contain an isometric copy of c. But $x = (1, 1, \dots, 1, \dots)$ is an extremal point of B_{W_f}

(because it is an extremal point of B_c). This shows that Zippin's result is false (see Fact 1).

It was this unexpected discovery that led us to the completely unknown to us polyhedral theory. It should be emphasized here that the main result in Zippin's paper is correct and states that every infinite-dimensional L_1-predual contains an isometric copy of the space c_0. On the other hand, the result described in Fact 1 is only one of the final comments in his work. The problem is that soon after, this result was used by others. Let's start from the beginning. In 1964, Lindenstrauss formulated the following

Theorem 4.3 (J. Lindenstrauss, [58])**.** *Let X be a Banach space. Consider the following properties.*

(1) $X^* = L_1(\mu)$ and X is a polyhedral space.
(2) For any Banach spaces $Y \subset Z$ and every compact operator $T : Y \to X$ there exists a compact extension $\tilde{T} : Z \to X$ with $\left\|\tilde{T}\right\| = \|T\|$.

Then the following implication holds: (2) $\Rightarrow$ (1).

J. Lindenstrauss posed the question whether (1) $\Rightarrow$ (2)? The answer was given a few years later by A. J. Lazar [50]. He used for this purpose the geometric characterization of this property based on the concept of w^*-closed *face* of the ball: recall that a closed and convex subset F of B_X is named a *face*, if $(1-\lambda)x + \lambda y \in F$ with $x, y \in B_X$ and $\lambda \in (0,1)$ imply $x, y \in F$. Moreover, we say that a face F is *proper* if $F \neq B_X$.

Fact 2. In the paper [*A. J. Lazar, Polyhedral Banach spaces and extensions of compact operators, Israel J. Math. 7 (1969), 357-364.*] Theorem 3 states that for every Lindenstrauss space X the following properties are equivalent:

(1) X is a polyhedral space;
(2) X does not contain an isometric copy of c;
(3) there are no infinite-dimensional w^*-closed proper faces of B_{X^*};
(4) for any Banach spaces $Y \subset Z$ and every compact operator $T : Y \to X$ there exists a compact extension $\tilde{T} : Z \to X$ with $\left\|\tilde{T}\right\| = \|T\|$.

The property of X described in (4) will be called *the compact norm-preserving extension property for compact operators*. The proof of the above theorem ran as follows: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1), and the result of Zippin mentioned earlier was used in the proof of the implication (2) $\Rightarrow$ (3).

Remark 6 ([CMPV2016], Section 3)**.** The set

$$S^+ = \left\{ (x(1), x(2), \dots) \in \ell_1 : \sum_{i=1}^{\infty} x(i) = 1, x(i) \geq 0, i = 1, 2, \dots \right\}$$

is an infinite-dimensional, $\sigma(\ell_1, W)$-compact proper face of B_{ℓ_1}. Therefore, the implication (2) $\Rightarrow$ (3) is false.

Lazar's result was then used by A. Gleit and R. McGuigan [26]:

Fact 3. In the paper [*A. Gleit, R. McGuigan, A note on polyhedral Banach spaces, Proc. Amer. Math. Soc. 33 (1972), 398-404.*], Theorem 1.2 states that for every Lindenstrauss space X the following properties are equivalent:

(1) $x^*(x) < 1$ for every $x \in S_X$ and $x^* \in (\mathrm{ext}(B_{X^*}))'$ (property (GM));
(2) X is a polyhedral space;
(3) X does not contain an isometric copy of c.

The proof of the above theorem was as follows: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1), and the proof of the implication (3) $\Rightarrow$ (1) was based on the (false) implication (2) $\Rightarrow$ (3) in Lazar's theorem.

Remark 7 ([CMPV2016], Section 3)**.** The space W fails property (GM). Indeed, in view of Theorem 2.3, for $x = (1, 1, \dots, 1, \dots) \in S_W$ and $x^* = e^* = \left(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots\right) \in (\mathrm{ext}(B_{W^*}))'$ we have $x^*(x) = 1$. This shows that the implication (3) $\Rightarrow$ (1) is false.

Fact 4. In the paper [*A. Gleit, R. McGuigan, A note on polyhedral Banach spaces, Proc. Amer. Math. Soc. 33 (1972), 398-404.*], Corollary 2.7 states that for every simplex space $A(K)$, i.e. the space of all affine continuous functions on a Choquet simplex K with the supremum norm, the following properties are equivalent:

(1) there is no $x^* \in (\mathrm{ext}(B_{X^*}))'$ with $\|x^*\| = 1$;
(2) X is a polyhedral space;
(3) X does not contain an isometric copy of c.

The proof of the above corollary ran as follows: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

Remark 8 ([CMPV2016], Section 3)**.** The space W is isometrically isomorphic to $A(S^+)$. This proves that the implication (3) $\Rightarrow$ (1) is false.

Remark 9 ([CMPV2016], Section 3)**.** The space W is a polyhedral space. To prove this assertion, it is enough to check that W enjoys the following condition:

$$\sup\{x^*(x) : x^* \in \mathrm{ext}(B_{X^*}) \setminus D(x)\} < 1$$

for every $x \in S_X$, where $D(x) = \{x^* \in S_{X^*} : x^*(x) = 1\}$. This condition was introduced by B. Brosowski and F. Deutsch [15]. Afterwards, R. Durier and P. L. Papini proved that it implies polyhedrality in the general case of Banach spaces ([21], Theorem 1).

By Remark 9 and Remark 7, the proof of the implication (3) $\Rightarrow$ (2) in Fact 3 is incorrect. It turns out, however, that the implication is true. Let us start with the ℓ_1-preduals case:

Theorem 4.4 ([CMPV2016], Theorem 4.1)**.** *Let X be a predual of ℓ_1. The following properties are equivalent:*

(1) *X is a polyhedral space;*
(2) *X does not contain an isometric copy of c;*
(3) $\sup\{x^*(x) : x^* \in \mathrm{ext}\,(B_{X^*}) \setminus D(x)\} < 1$ *for every $x \in S_X$ (property* (BD)*).*

Clearly, it is enough to prove that (2) $\Rightarrow$ (3). The main tool in the proof of this implication is a result of I. Gasparis ([25], Theorem 1.1).

In the general case of Lindenstrauss spaces, we have the following

Theorem 4.5 ([CMPV2016], Theorem 4.3)**.** *Let X be a Lindenstrauss space. The following properties are equivalent:*

(1) *X is a polyhedral space;*
(2) *X does not contain an isometric copy of c;*
(3) *X has the property (BD).*

The proof of the above theorem is based on Theorem 4.4, the paper [77], and a series of other known results that can be found in the book of Lacey [49].

Recall that the chain of implications in Lazar's theorem (see Fact 2) was interrupted. Therefore, the question posed by Lindenstrauss whether polyhedrality implies the compact norm-preserving extension property for compact operators still remains open. The answer is included in the following theorem:

Theorem 4.6 ([CMPV2016], Theorem 5.1). *Let X be a Banach space. Suppose that there exists $x \in S_X$ such that $D(x)$ is not norm-compact. Then there exist a separable Banach space Z, a complemented subspace $Y \subset Z$, and a compact operator $T\colon Y \to X$ such that T does not admit any compact extension $\tilde{T}\colon Z \to X$ of the same norm.*

Remark 10 ([CMPV2016], Section 5). For the space W and $x = (1, 1, \dots, 1, \dots) \in S_W$ we have $D(x) = S^+$ (see Theorem 2.3). Therefore, by Theorem 4.6, W fails the compact norm-preserving extension property for compact operators. By Remark 9 we conclude that the implication (1) $\Rightarrow$ (4) in Lazar's theorem is false. It means that the answer to the question posed by Lindenstrauss is negative.

The following result gives a necessary and sufficient condition for a Banach space to have the compact norm-preserving extension property for compact operators.

Theorem 4.7 ([CMPV2016], Theorem 5.3). *Let X be an infinite-dimensional Banach space. The following properties are equivalent.*

(1) *X is a Lindenstrauss space such that each set $D(x)$ ($x \in S_X$) is finite-dimensional (property Δ in* [24]*).*
(2) *For any Banach spaces $Y \subset Z$, every compact operator $T\colon Y \to X$ admits a compact norm-preserving extension $\tilde{T}\colon Z \to X$.*

It turns out that unlike polyhedrality, the property (3) in Lazar's theorem as well as the property (GM) in Gleit and McGuigan's theorem are equivalent to the compact norm-preserving extension property for compact operators and it is a simple consequence of another result of Lazar (Proposition 1 in [50]). We will return to this issue in Chapter 6.

Note. It is worth mentioning here that Professor Mordecay Zippin in [*Correction to "On some subspaces of Banach spaces whose duals are L_1 spaces"*, Proc. Amer. Math. Soc. (2018) DOI: 10.1090/proc/14196] has recently presented two correct versions of that Remark A and a short proof of his 1969 main result. The hyperplanes in c play the key role in his considerations.

5. Stability of the weak* fixed point property in ℓ_1

In this chapter we will characterize all separable Lindenstrauss spaces X such that X^* has the stable weak* fixed point property. In addition, for each of them we give the exact value of the stability constant for the weak* fixed point property.

Generally speaking, the issue of stability of the fixed point property deals with the following question: suppose that a Banach space X has the fixed point property; is this property preserved for spaces with "small" Banach-Mazur distance from X? Recall that the Banach-Mazur distance between isomorphic Banach spaces X and Y is defined by

$$d(X, Y) = \inf \left\{ \|\phi\| \left\|\phi^{-1}\right\| : \phi \text{ is an isomorphism from } X \text{ onto } Y \right\}.$$

We will now quote some known results devoted to the stability of fixed point property.

- The spaces $L_p(\mu)$ have the stable fixed point property for $p \in (1, \infty)$. The first result of this type comes from K. Goebel and W. A. Kirk [30] and it concerns the fixed point property for *uniformly lipschitzian mappings*: recall that a mapping $T : C \to C$ defined on a nonempty subset C of X is called *uniformly k-lipschitzian* if

$$\|T^n x - T^n y\| \le k \|x - y\|$$

holds for all $x, y \in C$ and for all $n \in \mathbb{N}$. The authors proved that if X is a uniformly convex Banach space with the modulus of convexity δ_X, C is a nonempty bounded closed and convex subset of X, and γ is the constant satisfying the equation

$$\gamma\left(1-\delta_X\left(\frac{1}{\gamma}\right)\right)=1,$$

then for every $k<\gamma$, every uniformly k-lipschitzian mapping $T:C\to C$ has a fixed point in C. Clearly, for every uniformly convex Banach space we have $\gamma>1$, and in the case of Hilbert spaces $\gamma=\frac{\sqrt{5}}{2}$. Later this result has been improved and generalized to the case of metric spaces by E. A. Lifshitz [53]. He proved that in the case of a Hilbert space the thesis of the above theorem remains true for the constant $k<\sqrt{2}$. It is not known whether this estimate is sharp. The well-known example of J. B. Baillon [5] shows that it can not exceed $\frac{\pi}{2}$. Observe that if for some $\gamma>1$ a Banach space X has the fixed point property for uniformly k-lipschitzian mappings with $k<\gamma$, then Y has the FPP whenever $d(X,Y)<\gamma$.

- If X is a Banach space such that $d(X,\ell_p)<\left(1+2^{\frac{1}{p-1}}\right)^{\frac{p-1}{p}}$ for some $1<p<\infty$, then X has the FPP (T. Domìnguez Benavides, [9]). In particular, if $p=2$, then we obtain estimate $d(X,\ell_2)<\sqrt{3}$. This result has been improved by Pei-Kee Lin [55]. He proved that if X is a Banach space isomorphic to a Hilbert space H such that $d(X,H)<\sqrt{\frac{5+\sqrt{13}}{2}}$, then X has the FPP. Subsequently, this estimate has been slightly improved by E. Mazcuñán-Navarro [62]. The author showed that the conclusion of the above theorem remains true under the assumption $d(X,H)<\sqrt{\frac{5+\sqrt{17}}{2}}$. However, the fundamental question about the existence of a reflexive space not having the FPP is still open.
- If X is a Banach space such that $d(X,c_0)<2$ or $d(X,c)<2$, then X has the w-FPP (J. Borwein, B. Sims, [13]). It is not known if the above estimates are sharp.
- If Y is a Banach space such that $d(\ell_1,Y)<2$, then Y has the w^*-FPP (P. M. Soardi, [76]). Moreover, the example of T. C. Lim [54] shows that this estimate is exact.

Although in Soardi's theorem mentioned above it is not clearly emphasized, the author assumes that the space ℓ_1 as well as Y are endowed with the weak* topology generated by c_0. Since Y is not necessary a dual space, the above approach does not allow one to consider a true w^*-FPP. In order to avoid this undesirable feature we decided to introduce a different definition of stability for the w^*-FPP (see [CMPP2018], Definition 3.2): we say that the dual space X^* has the *stable* $\sigma(X^*,X)$-FPP if there exists $\gamma>1$ such that Y^* has the $\sigma(Y^*,Y)$-FPP whenever $d(X,Y)<\gamma$.

Recall that separable Lindenstrauss spaces whose duals enjoy the weak* fixed point property have been characterized in Theorem 3.7. We will quote it here in a slightly changed form that is more suitable in the context of polyhedral properties to which we will come back in the next chapter:

Theorem 5.1 ([CMPP2018], Theorem 2.1)**.** *Let X be a separable Lindenstrauss space. Then the following are equivalent:*

(i) X^* has the $\sigma(X^*,X)$-FPP.
(ii) There is no infinite set $C\subset \operatorname{ext}(B_{X^*})$ such that $\overline{\operatorname{conv}(C)}^{*}\subset S_{X^*}$.

We know that if X is a separable Lindenstrauss space such that X^* is nonseparable, then X^* fails the w^*-FPP. Therefore, the only case we are interested in are preduals of ℓ_1. The main result in article [CMPP2018] is the following

Theorem 5.2 ([CMPP2018], Theorem 3.5)**.** *Let X be a predual of ℓ_1. Then the following are equivalent:*

(i) *ℓ_1 has the stable $\sigma(\ell_1, X)$-FPP.*
(ii) $(\mathrm{ext}(B_{\ell_1}))' \subset rB_{\ell_1}$ *for some* $0 \leq r < 1$.

It should be noted here that the proofs of Theorems 5.2 and 3.7 required using two completely different techniques.

The main idea in the proof of the implication $(i) \Rightarrow (ii)$ in Theorem 5.2 is modeling the shape of the ball in a way such that, without an explicit formula for a norm, we are able to conclude that some fixed point free mapping is nonexpansive. The main tools in the proof of the implication $(ii) \Rightarrow (i)$ are some techniques developed by Soardi in [76] and the following

Lemma 5.3 ([CMPP2018], Lemma 3.3)**.** *Let X be a predual of ℓ_1.*

(a) *For every sequence $\{x_n^*\} \subset \ell_1$ coordinatewise converging to x_0^* and such that $\lim_{n\to\infty} \|x_n^* - x_0^*\|$ exists, we have*

$$\lim_{m\to\infty} \lim_{n\to\infty} \|x_n^* - x_m^*\| = 2 \lim_{n\to\infty} \|x_n^* - x_0^*\| .$$

If, in addition, $(\mathrm{ext}(B_{\ell_1}))' \subset rB_{\ell_1}$ *for some* $0 \leq r < 1$, *then*

(b) *for every sequence $\{x_n^*\} \subset \ell_1$ that is $\sigma(\ell_1, X)$-convergent to x^* and coordinatewise convergent to 0, we have*

$$\|x^*\| \leq r \liminf_{n\to\infty} \|x_n^*\| ;$$

(c) *for every sequence $\{x_n^*\} \subset \ell_1$ that is $\sigma(\ell_1, X)$-convergent to x^*, up to a subsequence we have*

$$\lim_{n\to\infty} \|x_n^* - x^*\| \leq \frac{1+r}{2} \lim_{m\to\infty} \lim_{n\to\infty} \|x_n^* - x_m^*\| .$$

We will now discuss the quantitative aspects of the stable weak* fixed point property. For this purpose, for every Banach space X such that X^* has the $\sigma(X^*, X)$-FPP, we introduce the following stability constant:

$$\gamma^*(X) = \sup\left\{\gamma > 0 : d(X, Y) \leq \gamma \Rightarrow Y^* \text{ has the } \sigma(Y^*, Y)\text{-FPP } \right\}.$$

As we will see in a moment, for a given predual X of ℓ_1 the value of the constant $\gamma^*(X)$ depends only on the radius $r^*(X)$ of the smallest ball containing all $\sigma(\ell_1, X)$-cluster points of the set of extreme points of B_{ℓ_1}:

$$r^*(X) = \inf\left\{r > 0 : (\mathrm{ext}(B_{\ell_1}))' \subset rB_{\ell_1}\right\}.$$

The proof of the implication $(ii) \Rightarrow (i)$ in Theorem 5.2 shows that if $d(Y, X) < \frac{2}{1+r}$, then Y^* has the $\sigma(Y^*, Y)$-FPP. Consequently,

$$\gamma^*(X) \geq \frac{2}{1 + r^*(X)}.$$

Moreover, if $r^*(X) = 1$, then, using the implication $(i) \Rightarrow (ii)$ in Theorem 5.2, we obtain $\gamma^*(X) = 1$. If $r^*(X) = 0$, then X is isometric to c_0. Therefore, the example of T. C. Lim [54] shows that in this case $\gamma^*(X) = 2$.

As we did in the paper [CMPP2017] we will now change the notation for hyperplanes in the space c. Namely, for every $e^* = (e^*(1), e^*(2), \dots) \in B_{\ell_1}$ we put

$$W_{e^*} = \left\{ x = (x(1), x(2), \dots) \in c : \lim_{i\to\infty} x(i) = \sum_{i=1}^{\infty} e^*(i)x(i) \right\}.$$

The set W_{e^*} is a hyperplane in c,

$$W_{e^*} = \{x \in c : f(x) = 0\} = \left\{ x \in c : f(1) \lim_{i\to\infty} x(i) + \sum_{i=1}^{\infty} f(i+1)x(i) = 0 \right\},$$

where

$$f = \left(-\frac{1}{1+\|e^*\|_{\ell_1}}, \frac{e^*(1)}{1+\|e^*\|_{\ell_1}}, \frac{e^*(2)}{1+\|e^*\|_{\ell_1}}, \dots, \frac{e^*(i)}{1+\|e^*\|_{\ell_1}}, \dots \right) \in S_{\ell_1}.$$

By Theorem 2.3,

$$e_n^* \overset{\sigma(\ell_1, W_{e^*})}{\longrightarrow} e^*.$$

The main result in the paper [CMPP2017] are Propositions 5.4, 5.5, and 5.6 presented below. Here $\|\cdot\|_\infty$ and $|\cdot|_{\ell_1}$ denote the standard norm in c and ℓ_1, respectively.

Proposition 5.4 ([CMPP2017], Proposition 2.2)**.** *Let* $e^* = (e^*(1), \dots, e^*(n), 0, 0, \dots) \in \ell_1$ *and* $r_n := |e^*|_{\ell_1} \in (0,1)$*. For all* $x \in W_{e^*}$*, define*

$$\|x\|_n = \left(\left\|R_n x^+\right\|_\infty \vee r_n \left\|R_n x^-\right\|_\infty + \left\|R_n x^-\right\|_\infty \vee r_n \left\|R_n x^+\right\|_\infty \right) \vee (1+r_n) \|P_n x\|_\infty .$$

Then

$$(W_{e^*}, \|\cdot\|_n)^* = (\ell_1, |\cdot|_n),$$

where

$$|f|_n = \max\left\{ \frac{r_n |R_n f^+|_{\ell_1} + |R_n f^-|_{\ell_1}}{1+r_n}, \frac{|R_n f^+|_{\ell_1} + r_n |R_n f^-|_{\ell_1}}{1+r_n} \right\} + \frac{|P_n f|_{\ell_1}}{1+r_n},$$

and a duality map $\phi : (\ell_1, |\cdot|_n) \to (W_{e^*}, \|\cdot\|_n)^*$ *is defined by:*

$$(\phi(f))(x) = \sum_{j=1}^{+\infty} x(j) f(j),$$

where $f = (f(1), f(2), \dots) \in \ell_1$ *and* $x = (x(1), x(2), \dots) \in W_{e^*}$*.*

Proposition 5.5 ([CMPP2017], Proposition 2.3)**.** *Let* $e^* = (e^*(1), \dots, e^*(n), 0, 0, \dots) \in \ell_1$ *with* $r_n := |e^*|_{\ell_1} \in (0,1)$*. Then* $(W_{e^*}, \|\cdot\|_n)^* = (\ell_1, |\cdot|_n)$ *lacks the* w^**-FPP.*

Proposition 5.6 ([CMPP2017], Proposition 2.4)**.** *If* X *is a predual of* ℓ_1 *with* $r^*(X) \in (0,1)$*, then* $\gamma^*(X) \le \frac{2}{1+r^*(X)}$*.*

In the proof of the main result, the following technical lemma, resulting from the Ostrovskiy̆'s paper [67], was also useful (it allowed us to shorten the proof of the main result by omitting the explicit formulas for isomorphisms):

Lemma 5.7 ([CMPP2017], Lemma 2.1)**.** *Let* $\{x_n^*\} \subset X^*$ *be a sequence norm convergent to* x^**. Then*

$$\lim_{n\to\infty} d(\ker x^*, \ker x_n^*) = 1.$$

Finally, taking into account Theorem 5.2 and Proposition 5.6 (in tandem with Propositions 5.4 and 5.5), we obtain the exact value of the constant $\gamma^*(X)$:

Theorem 5.8 ([CMPP2018], Theorem 3.5 oraz [CMPP2017], Proposition 2.4). *Let X be a predual of ℓ_1. If ℓ_1 has the $\sigma(\ell_1, X)$-FPP, then*

$$\gamma^*(X) = \frac{2}{1 + r^*(X)}.$$

It should be emphasized here that the technique used in the proof of the implication $(i) \Rightarrow (ii)$ in Theorem 5.2 is completely different from the one presented in the proofs of Propositions 5.4-5.6.

In the further part of the paper [CMPP2017] we consider stability of the weak* fixed point property in the restricted framework of ℓ_1-preduals. Namely, for a predual X of ℓ_1 that has the $\sigma(\ell_1, X)$-FPP, we are interested in the estimate of the following constant:

$$\eta^*(X) = \sup \{\eta > 0 : \, Y^* = \ell_1, \, d(X, Y) \leq \eta \Rightarrow Y^* \text{ has the } \sigma(\ell_1, Y)\text{-FPP}\}.$$

We obtained the exact value of the constant $\eta^*(X)$ in the case when $X = c_0$. Let us notice first that $\eta^*(c_0) \leq 3$, which is an immediate consequence of the result of M. Cambern [19] which states that $d(c_0, c) = 3$. Since for every ℓ_1-predual X we have $\gamma^*(X) \leq \eta^*(X)$, it follows that $2 \leq \eta^*(c_0) \leq 3$. It turns out that $\eta^*(c_0) = 3$:

Theorem 5.9 ([CMPP2017], Theorem 3.7). *Let X be a predual of ℓ_1 isomorphic to c_0. Suppose that X^* fails the w^*-FPP. If $T : X \to c_0$ is an isomorphism with $\|T^{-1}\| = 1$, then $\|T\| \geq 3$.*

In the proof of the above result, we used Theorem 3.7, Proposition 3.5 and the following

Proposition 5.10 ([CMPP2017], Prop. 3.4). *Let X be a Banach space containing an isometric copy of c and let $T : c_0 \to X$ be an onto linear operator with $\|T\| = 1$. If $\tilde{T} : X/\mathrm{ker}T \to Y$ denotes a mapping defined by $T = \tilde{T}\pi$, where $\pi : X \to X/\mathrm{ker}T$ is the quotient map, then $\|\tilde{T}^{-1}\| \geq 3$.*

Furthermore, in the proof of Proposition 5.10 a key role was played by the result of D. E. Alspach [1] and Y. Gordon ([37], Theorem 2.1).

There are many ℓ_1-preduals X such that ℓ_1 has the $\sigma(\ell_1, X)$-FPP and $d(X, c_0) = 3$. In the example we will present here, we will use the following technical result:

Proposition 5.11 ([CMPP2017], Prop. 3.8). *If $\alpha \in S_{\ell_1}$, then $d(c_0, W_\alpha) = 3$.*

Example 4. Let $\alpha = \left(-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, \dots\right) \in S_{\ell_1}$. Then the space W_α is an ℓ_1-predual and $d(c_0, W_\alpha) = 3$ (Proposition 5.11). Moreover, ℓ_1 enjoys the $\sigma(\ell_1, W_\alpha)$-FPP (Proposition 3.1).

Analysis of the proof of Proposition 5.11 shows that $d(W_\alpha, c_0) \leq 1 + 2\,|\alpha|_{\ell_1}$ for every $\alpha \in B_{\ell_1}$. This inequality, Proposition 5.11 and Theorem 5.2 allow us to characterize the stability constant $\gamma^*(W_\alpha)$ in terms of the Banach-Mazur distance $d(W_\alpha, c_0)$:

Corollary 5.12 ([CMPP2017], Corollary 3.10). *Let $\alpha \in B_{\ell_1}$ be such that ℓ_1 has the $\sigma(\ell_1, W_\alpha)$-FPP. Then $\gamma^*(W_\alpha) > 1$ if and only if $d(W_\alpha, c_0) < 3$.*

6. Geometric equivalences for polyhedral properties in the setting of ℓ_1-preduals

After the publication of Klee's work, polyhedrality was extensively studied and several different definitions have been stated. In the framework of Lindenstrauss spaces X we can distinguish the following:

- (pol-i): $(\mathrm{ext}(B_{X^*}))' \subset \{0\}$ ([60]);
- (pol-ii): $(\mathrm{ext}(B_{X^*}))' \subset rB_{X^*}$ for some $0 < r < 1$ ([24]);
- (pol-iii): $(\mathrm{ext}(B_{X^*}))' \subset \mathrm{int}(B_{X^*})$ ([24]);
- (pol-iv): there is no infinite set $C \subset \mathrm{ext}(B_{X^*})$ such that $\overline{\mathrm{conv}(C)}^* \subset S_{X^*}$ (property (pol-iii) in [CMPP2018]);
- (pol-v): there is no infinite-dimensional w^*-closed proper face of B_{X^*} ([50]);
- (pol-vi): $x^*(x) < 1$ whenever $x \in S_X$ and $x^* \in (\mathrm{ext}(B_{X^*}))'$ ([26]);
- (pol-vii): $\mathrm{ext}(D(x))$ is finite for each $x \in S_X$ (property (Δ) in [24]);
- (pol-viii): $\sup\{x^*(x) : x^* \in \mathrm{ext}(B_{X^*}) \setminus D(x)\} < 1$ for each $x \in S_X$ ([15]);
- (pol-K): the unit ball of every finite-dimensional subspace of X is a polytope ([47]).

In addition, we will consider the following properties of X and its dual X^*:

- w^*-NS: X^* has the *weak* normal structure* (briefly, $\sigma(X^*, X)$-NS or w^*-NS) if every convex, $\sigma(X^*, X)$-compact set $C \subset X^*$ with positive diameter contains a point which is *not diametral*, that is, there exists $x^* \in C$ such that
$$\sup\{\|x^* - y^*\| : y^* \in C\} < \mathrm{diam}(C) := \sup\{\|x^* - y^*\| : x^*, y^* \in C\}$$
(see [14] and [72]).
- w^*-KK: X^* has the *weak* Kadec-Klee property* (briefly, $\sigma(X^*, X)$-KK or w^*-KK) if for every sequence (x_n^*) in S_{X^*} which is $\sigma(X^*, X)$-convergent to $x^* \in S_{X^*}$, we have $\lim\limits_{n\to\infty} \|x_n^* - x^*\| = 0$.
- w^*-UKK: X^* has the *uniform weak* Kadec-Klee property* (briefly, $\sigma(X^*, X)$-UKK or w^*-UKK) provided that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if a sequence (x_n^*) in B_{X^*} is $\sigma(X^*, X)$-convergent to x^* and
$$\mathrm{sep}\{x_n^*\} := \inf\{\|x_n^* - x_m^*\| : n \neq m\} > \varepsilon,$$
then $\|x^*\| < 1 - \delta$.
- w^*-GGLD: X^* has the *weak* Generalized Gossez-Lami Dozo property* (briefly, $\sigma(X^*, X)$-GGLD or w^*-GGLD) if
$$\lim_{n\to\infty} \|x_n^*\| < \lim_{n,m;n\neq m} \|x_n^* - x_m^*\|$$
for every $\sigma(X^*, X)$-null sequence (x_n^*) in X^* such that both limits exist and $\lim\limits_{n\to\infty} \|x_n^*\| \neq 0$ (see [40] and [[10], Definition 3]).
- w^*-O: X^* satisfies the *weak* Opial property* (briefly, $\sigma(X^*, X)$-O or w^*-O) if
$$\liminf_{n\to\infty} \|x_n^*\| < \liminf_{n\to\infty} \|x^* + x_n^*\|$$
for every $\sigma(X^*, X)$-null sequence (x_n^*) in X^* and every $x^* \neq 0$ (see [66] and [72]).
- w^*-UO: X^* has the *uniform weak* Opial property* (briefly, $\sigma(X^*, X)$-UO or w^*-UO) if for every $c > 0$ there exists $r > 0$ such that
$$1 + r \leq \liminf_{n\to\infty} \|x^* + x_n^*\|$$
for each $x^* \in X^*$ with $\|x^*\| \geq c$ and each $\sigma(X^*, X)$-null sequence (x_n^*) in X^* such that $\liminf\limits_{n\to\infty} \|x_n^*\| \geq 1$ (see [71] and [72]).
- F.N.E.P.: X has the *finite-dimensional norm-preserving extension property* for bounded operators with a finite-dimensional range (briefly, F.N.E.P.) if for any Banach spaces $Y \subset Z$ and every bounded operator $T : Y \to X$ with $\dim(T(Y)) < \infty$ there exists an extension $\widetilde{T} : Z \to X$ with $\left\|\widetilde{T}\right\| = \|T\|$ and $\dim(\widetilde{T}(Z)) < \infty$ (see [58]).

- C.N.E.P.: X has the *compact norm-preserving extension property* for compact operators (briefly, C.N.E.P) if for any Banach spaces $Y \subset Z$ and every compact operator $T : Y \to X$ there exists a compact extension $\widetilde{T} : Z \to X$ with $\left\|\widetilde{T}\right\| = \|T\|$ (see [58]).

Let X be a predual of ℓ_1. The diagram below shows all the relations between different polyhedral properties of X, C.N.E.P, F.N.E.P. as well as the w^*-NS, the w^*-FPP, the w^*-GGLD, the w^*-KK, the stable w^*-FPP, the w^*-UKK, the w^*-O and the w^*-UO in the dual space.

$$\begin{tikzcd}
\sigma(\ell_1,X)\text{-O} \arrow[d, Leftrightarrow] & \text{stable } \sigma(\ell_1,X)\text{-FPP} \arrow[d, Leftrightarrow] & \sigma(\ell_1,X)\text{-GGLD} \arrow[d, Leftrightarrow] & \sigma(\ell_1,X)\text{-FPP} \arrow[d, Leftrightarrow] & \text{C.N.E.P.} \arrow[d, Leftrightarrow] & \text{F.N.E.P.} \arrow[d, Leftrightarrow] \\
\text{(pol-i)} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & \text{(pol-ii)} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & \text{(pol-iii)} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & \text{(pol-iv)} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & \text{(pol-v)} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & \text{(pol-viii)} \arrow[d, Leftrightarrow] \\
\sigma(\ell_1,X)\text{-UO} \arrow[d, Leftrightarrow] & \sigma(\ell_1,X)\text{-UKK} & \sigma(\ell_1,X)\text{-KK} & \sigma(\ell_1,X)\text{-NS} & \text{(pol-vi)} \arrow[d, Leftrightarrow] & \text{(pol-K)} \arrow[d, Leftrightarrow] \\
X = c_0 & & & & \text{(pol-vii)} & c \nsubseteq X
\end{tikzcd}$$

The implications $(X = c_0) \Rightarrow \sigma(\ell_1, X)$-UO $\Rightarrow \sigma(\ell_1, X)$-O $\Rightarrow$ (pol-i) $\Rightarrow$ (pol-ii) $\Rightarrow$ (pol-iii) $\Rightarrow$ (pol-iv) $\Rightarrow$ (pol-v) and (pol-vi) $\Rightarrow$ (pol-vii) are easy to prove. The implication (pol-i) $\Rightarrow$ $(X = c_0)$ follows from ([21], Proposition 2). The implication (pol-K) $\Rightarrow$ $(c \nsubseteq X)$ follows from the fact that c is not a polyhedral space. The equivalence F.N.E.P. $\Leftrightarrow$ (pol-K) is proved in ([58], Proposition 2). The implication (pol-v) $\Rightarrow$ C.N.E.P. is proved in ([50], Theorem 3). The implication (pol-viii) $\Rightarrow$ (pol-K) holds for any Banach space (see [21], Theorem 1). The implication (pol-v) $\Rightarrow$ (pol-vi) follows from ([26], Theorem 1.2). The implication C.N.E.P. $\Rightarrow$ (pol-K) is proved in [58]. The implication $\sigma(\ell_1, X)$-FPP $\Rightarrow$ (pol-iv) follows from [39].

The implication (pol-iv) $\Rightarrow$ $\sigma(\ell_1, X)$-FPP holds by Theorem 3.7 (see also Theorem 5.1). The equivalence stable $\sigma(\ell_1, X)$-FPP $\Leftrightarrow$ (pol-ii) holds by Theorem 5.2 and Theorem 5.8. The implications $(c \nsubseteq X) \Rightarrow$ (pol-K) and (pol-K) $\Rightarrow$ (pol-viii) hold by Theorem 4.4. The implication (C.N.E.P.) $\Rightarrow$ (pol-vii) follows from Theorem 4.7. The implication (pol-vii) $\Rightarrow$ (pol-v) follows easily from the following lemma.

Lemma 6.1 ([CMPP2018], Lemma 4.3)**.** *Let X be a Lindenstrauss space and let F be a w^*-closed proper face of B_{X^*}. Then there exists $x \in S_X$ such that $F \subset D(x)$.*

The equivalences $\sigma(\ell_1, X)$-UKK $\Leftrightarrow$ (pol-ii), $\sigma(\ell_1, X)$-NS $\Leftrightarrow$ (pol-iv) and $\sigma(\ell_1, X)$-KK $\Leftrightarrow$ (pol-iii) $\Leftrightarrow$ $\sigma(\ell_1, X)$-GGLD hold by Propositions 6.2-6.3 and Theorem 6.4 presented below.

Proposition 6.2 ([P2018], Proposition 2.1)**.** *Let X be a predual of ℓ_1. Then the following are equivalent:*

(1) ℓ_1 has the $\sigma(\ell_1, X)$-UKK property.
(2) $(\mathrm{ext}(B_{\ell_1}))' \subseteq rB_{\ell_1}$ for some $0 < r < 1$.

Proposition 6.2 and Theorem 5.2 show that for the space ℓ_1 with predual X, the stable $\sigma(\ell_1, X)$-FPP is equivalent to the $\sigma(\ell_1, X)$-UKK.

Proposition 6.3 ([P2018], Proposition 2.2). *Let X be a predual of ℓ_1. Then the following are equivalent:*

(1) ℓ_1 has the $\sigma(\ell_1, X)$-NS.
(2) There is no infinite set $C \subset \operatorname{ext}(B_{\ell_1})$ such that $\overline{\operatorname{conv}(C)}^* \subset S_{\ell_1}$.

Proposition 6.3 and Theorem 3.7 (see also Theorem 5.1) show that for the space ℓ_1 with predual X, the $\sigma(\ell_1, X)$-FPP is equivalent to the $\sigma(\ell_1, X)$-NS.

Theorem 6.4 ([P2018], Theorem 2.3). *Let X be a predual of ℓ_1. Then the following are equivalent:*

(1) ℓ_1 has the $\sigma(\ell_1, X)$-KK property.
(2) ℓ_1 has the $\sigma(\ell_1, X)$-GGLD property.
(3) $(\operatorname{ext}(B_{\ell_1}))' \subset \operatorname{int}(B_{\ell_1})$.
(4) For every sequence (x_n^*) in ℓ_1 that is $\sigma(\ell_1, X)$-convergent to x^* and coordinatewise convergent to 0, and $\liminf_{n\to\infty} \|x_n^*\| > 0$, we have

$$\|x^*\| < \liminf_{n\to\infty} \|x_n^*\| .$$

Theorem 6.4 will play an important role in the examples 9-10 discussed in Chapter 7.

We will end this chapter with examples showing that none of the one-way implications in our diagram can be reversed. For this purpose, we will use the hyperplanes in c discussed earlier: for $\alpha = (\alpha(1), \alpha(2), \dots) \in B_{\ell_1}$ let

$$W_\alpha = \left\{ x = (x(1), x(2), \dots) \in c : \lim_{i\to\infty} x(i) = \sum_{i=1}^{\infty} \alpha(i)x(i) \right\}.$$

Example 5 ([CMPP2018], Example 4.7). Let $\alpha = (r/2, r/2, 0, 0, \dots) \in \ell_1$ for $0 < r < 1$. Then (e_n^*) is $\sigma(\ell_1, W_\alpha)$-convergent to α. This shows that [(pol-ii) $\not\Rightarrow$ (pol-i)].

Example 6 ([CMPP2018], Example 4.8). Let $\alpha = (-1/2, -1/4, -1/8, \dots) \in \ell_1$. Then W_α has property (pol-iv), but (e_n^*) is $\sigma(\ell_1, W_\alpha)$-convergent to $\alpha \in S_{\ell_1}$. Therefore, [(pol-iv) $\not\Rightarrow$ (pol-iii)].

Example 7 ([CMPP2018], Example 4.9). Let $\alpha = (1/2, -1/4, 1/8, -1/16, \dots) \in \ell_1$. Then W_α satisfies property (pol-v). However, by considering the set $C = \{e_1^*, e_3^*, e_5^*, \dots\}$, it is easy to see that W_α fails property (pol-iv). Consequently, [(pol-v) $\not\Rightarrow$ (pol-iv)].

Example 8 ([CMPV2016], Section 3). Let $\alpha = (1/2, 1/4, 1/8, \dots) \in \ell_1$. Then W_α has property (pol-viii) but it lacks (pol-v). Therefore, [(pol-viii) $\not\Rightarrow$ (pol-v)].

Moreover, Examples 9 and 10 in Chapter 7 show that [(pol-iii) $\not\Rightarrow$ (pol-ii)].

7. On ℓ_1-preduals distant by 1

Recall that the dual X^* of X fails the stable $\sigma(X^*, X)$-FPP if for every $\varepsilon > 0$ there exists a Banach space Y such that Y^* lacks the $\sigma(Y^*, Y)$-FPP and $d(X, Y) \le 1 + \varepsilon$. Clearly, the w^*-FPP is isometrically invariant, that is, if X^* has the $\sigma(X^*, X)$-FPP, then the same is true for any dual Banach space Y^* whenever Y is isometric to X. However, the fact that $d(X, Y) = 1$ does not mean that X and Y are isometric and the first example of such spaces was given by Czesław Bessaga and Aleksander Pełczyński in [68]. Therefore, the following question arises:

Are there Banach spaces X and Y such that X^* has the $\sigma(X^*, X)$-FPP, Y^* fails the $\sigma(Y^*, Y)$-FPP and $d(X, Y) = 1$?

In fact, this problem has initiated a new research path associated with a more subtle approach to the general problem of lack of stability of geometric properties of Banach spaces (see [P2018]). Namely, we will say that a given property $\mathcal{P}$ *is not invariant under the Banach-Mazur distance* 1 if there exist two Banach spaces X and Y such that X enjoys property $\mathcal{P}$, Y fails this property and $d(X,Y) = 1$. Moreover, we will say that a given property $\mathcal{P}$ is *invariant under the Banach-Mazur distance* 1 if for each pair of Banach spaces X and Y with $d(X,Y) = 1$, X has property $\mathcal{P}$ if and only if Y has property $\mathcal{P}$.

The main results of [P2018] concern the case of separable Lindenstrauss spaces. First, in the framework of ℓ_1-preduals, we give some geometric equivalence for polyhedral properties (see Propositions 6.2-6.3 and Theorem 6.4 in Chapter 6). Then, we show that, even in the restricted setting of ℓ_1-preduals, most of the notions of polyhedral Banach spaces (labelled (pol-iii)-(pol-viii) and (pol-K)) and their geometric equivalences (see the diagram in Chapter 6) fail to be invariant under the Banach-Mazur distance 1 (see Examples 9-10).

In this context, the following will be useful

Lemma 7.1 (Lemma 3.1 in [P2018])**.** *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces that are isomorphic to a given Banach space X and satisfy $\lim_{n\to\infty} d(X_n, X) = 1$. Let*

$$Y = \left(\sum_{n=1}^{\infty} X_n\right)_{c_0} \qquad \text{and} \qquad Z = \left(X \oplus \sum_{n=2}^{\infty} X_{n-1}\right)_{c_0}.$$

Then $d(Y,Z) = 1$. Moreover, the same conclusion holds if we replace the c_0-direct sum by the ℓ_p-direct sum with $1 \le p \le \infty$.

Example 9 ([P2018], Example 3.2)**.** Let $x^* = (1, 0, 0, \dots) \in \ell_1$. We define the spaces X and Y as a c_0-direct sums of appropriate hyperplanes in the space c:

$$X = \left(\sum_{n=1}^{\infty} W_{\frac{n}{n+1}x^*}\right)_{c_0}, \quad Y = \left(W_{x^*} \oplus \sum_{n=2}^{\infty} W_{\frac{n-1}{n}x^*}\right)_{c_0}.$$

Then $X^* = \left(\sum_{n=1}^{\infty} \ell_1\right)_{\ell_1} = \ell_1$, $Y^* = \left(\ell_1 \oplus \sum_{n=2}^{\infty} \ell_1\right)_{\ell_1} = \ell_1$ and

$$(\operatorname{ext}(B_{X^*}))' \;=\; \{(0,0,\dots)\} \cup \bigcup_{n=1}^{\infty} \left\{ \pm(\underbrace{0,\dots,0}_{n-1}, \frac{n}{n+1}x^*, 0, 0, \dots) \right\} \subset \operatorname{int}(B_{\ell_1}).$$

Therefore, by Theorem 3.7, ℓ_1 has the $\sigma(\ell_1, X)$-FPP. However, we can say more. Namely, Theorem 6.4 shows that ℓ_1 has the $\sigma(\ell_1, X)$-KK property. Consequently, by using ([39], Theorem 5), ℓ_1 enjoys the $\sigma(\ell_1, X)$-FPP for *mappings of asymptotically nonexpansive type*; recall that a mapping $T : C \to C$ is of asymptotically nonexpansive type if T^N is continuous for some $N \in \mathbb{N}$ and

$$\limsup_{n\to\infty} (\sup \{\|T^n(x) - T^n(y)\| - \|x - y\| : y \in C\}) \le 0$$

for every $x \in C$. Clearly, every nonexpansive map is of asymptotically nonexpansive type. On the other hand, W_{x^*} is isometrically isomorphic to c and so Y fails the property (pol-K). Moreover, by Theorem 3.7, ℓ_1 lacks the $\sigma(\ell_1, Y)$-FPP for contractive mappings.

We will show now that $d(X,Y) = 1$. For this purpose, for every $n \in \mathbb{N}$ we define the mapping $\psi_n : W_{\frac{n}{n+1}x^*} \to W_{x^*}$ by

$$\psi_n\left(x(1), x(2), \dots\right) = \left(\frac{n}{n+1}x(1), x(2), x(3), \dots\right).$$

It is easy to check that every ψ_n is an isomorphism, $\lim_{n\to\infty}\|\psi_n\|\,\|\psi_n^{-1}\|=1$, and therefore $\lim_{n\to\infty} d(W_{\frac{n}{n+1}x^*}, W_{x^*})=1$. Consequently, by Lemma 7.1, we get $d(X,Y)=1$.

Example 10 ([P2018], Example 3.3)**.** Let $x_1^*=(1,0,0,\dots)$, $x_2^*=\left(\frac{1}{2},\frac{1}{4},\frac{1}{8},\dots\right)$, $x_3^*=\left(\frac{1}{2},-\frac{1}{4},\frac{1}{8},-\frac{1}{16},\dots\right)$ and $x_4^*=\left(\frac{1}{2},-\frac{1}{4},-\frac{1}{8},-\frac{1}{16},\dots\right)$. For $i=1,2,3,4$ we define

$$X_i=\left(\sum_{n=1}^{\infty} W_{\frac{n}{n+1}x_i^*}\right)_{c_0} \quad \text{and} \quad \widetilde{X_i}=\left(W_{x_i^*}\oplus\sum_{n=2}^{\infty} W_{\frac{n-1}{n}x_i^*}\right)_{c_0}.$$

Then $X_i^*=\left(\sum_{n=1}^{\infty}\ell_1\right)_{\ell_1}=\ell_1$ and $\widetilde{X_i}^*=\left(\ell_1\oplus\sum_{n=2}^{\infty}\ell_1\right)_{\ell_1}=\ell_1$ for $i=1,2,3,4$. Next, for every $n\geq 1$ we define the mapping $\psi_{i,n}: W_{\frac{n}{n+1}x_i^*}\to W_{x_i^*}$ by

$$\psi_{1,n}\left(x(1),x(2),\dots\right)=\left(\frac{n}{n+1}x(1),x(2),x(3),\dots\right)$$

and for $i=2,3,4$

$$\psi_{i,n}\left(x(1),x(2),\dots\right)=\left(x(1)-\frac{2}{n}\lim_{i\to\infty}x(i),x(2),x(3),\dots\right).$$

Then every mapping $\psi_{i,n}$ is an onto isomorphism, $\lim_{n\to\infty}\|\psi_{i,n}\|\left\|\psi_{i,n}^{-1}\right\|=1$, and therefore $\lim_{n\to\infty} d(W_{\frac{n}{n+1}x_i^*}, W_{x_i^*})=1$. Consequently, by Lemma 7.1, $d(X_i,\widetilde{X_i})=1$. Furthermore, for $i=1,2,3,4$ we have

$$\left(\operatorname{ext}(B_{X_i^*})\right)'=\{(0,0,\dots)\}\cup\bigcup_{n=1}^{\infty}\left\{\pm(\underbrace{0,\dots,0}_{n-1},\frac{n}{n+1}x_i^*,0,0,\dots)\right\},$$

$$\left(\operatorname{ext}(B_{\widetilde{X_i}^*})\right)'=\{(0,0,\dots)\}\cup\{\pm\left(x_i^*,0,0,\dots\right)\}\cup\bigcup_{n=2}^{\infty}\left\{\pm(\underbrace{0,\dots,0}_{n-1},\frac{n-1}{n}x_i^*,0,0,\dots)\right\}.$$

Let

$$Y_1=\left(\widetilde{X_1}\oplus X_2\oplus X_3\oplus X_4\right)_\infty, Y_2=\left(X_1\oplus\widetilde{X_2}\oplus X_3\oplus X_4\right)_\infty,$$
$$Y_3=\left(X_1\oplus X_2\oplus\widetilde{X_3}\oplus X_4\right)_\infty, Y_4=\left(X_1\oplus X_2\oplus X_3\oplus\widetilde{X_4}\right)_\infty,$$
$$Y_5=\left(X_1\oplus X_2\oplus X_3\oplus X_4\right)_\infty.$$

It is easy to check that the following statements are true (see the diagram in Chapter 6):

- $Y_i^*=\ell_1$ and $d(Y_i,Y_j)=1$ for $i,j=1,2,3,4,5$.
- Y_1 fails the property (pol-K) because $W_{x_1^*}$ is isometric to c.
- By Theorem 4.1 (see also Corollary 4.2), the space Y_2 does not contain an isometric copy of c. Therefore, Y_2 has the property (pol-K). However, Y_2 lacks the property (pol-v).
- Y_3 satisfies (pol-v) but it lacks (pol-iv).
- Y_4 satisfies (pol-iv) but it lacks (pol-iii).
- Y_5 satisfies (pol-iii).

The second part of the article [P2018] is devoted to the general case of Banach spaces. We indicate there some other geometric properties that are not invariant under the Banach-Mazur distance 1 and play an important role in *Mathematical Analysis*, in particular *Metric Fixed Point Theory*. Among them are: uniform convexity in every direction (UCED), locally uniform rotundity (LUR), smoothness, (weak*) Opial property, Kadec-Klee property, normal structure and the weak fixed point property for nonexpansive mappings. On the other hand, uniform convexity, uniform smoothness, uniform Opial property, and uniform Kadec-Klee property are invariant under the Banach-Mazur distance 1.

Moreover, since uniform weak* Opial property and uniform weak* Kadec-Klee property are invariant under the Banach-Mazur distance 1, therefore, in the restricted setting of preduals of ℓ_1, the polyhedral properties (pol-i) and (pol-ii) are invariant under the Banach-Mazur distance 1 (see the diagram in Chapter 6).

8. Discussion of other scientific and research achievements

My remaining scientific achievements are 11 papers and a book.

Papers published before Ph.D. degree

[GP2008] K. Goebel and Ł. Piasecki, *A new estimate for the optimal retraction constant*, Proceedings of the Second International Symposium on Banach and Function Spaces 2006, Kitakyushu, Japan, September 14-17, 2006, pages 77-83, Yokohama Publishers, 2008.

[P2009] Ł. Piasecki, *Retracting ball onto sphere in $BC_0(\mathbb{R})$*, Topol. Methods Nonlinear Anal. 33 (2009), 307-313.

[P2011] Ł. Piasecki, *Retracting a ball onto a sphere in some Banach spaces*, Nonlinear Anal. 74 (2011), 396-399.

[PP2011] V. Pérez-García and Ł. Piasecki, *Lipschitz constants for iterates of mean lipschitzian mappings*, Nonlinear Anal. 74 (16) (2011), 5643-5647.

[PP2012] V. Pérez-García and Ł. Piasecki, *On mean nonexpansive mappings and the Lifshitz constant*, J. Math. Anal. Appl. 396 (2) (2012), 448-454.

Papers and a book published after Ph.D. degree

[P2013] **Ł. Piasecki, *Classification of Lipschitz Mappings*, CRC Press, Taylor & Francis Group, A Chapman & Hall Book, 2013. (a book)**

[PP2013] V. Pérez-García and Ł. Piasecki, *Spectral radius for mean lipschitzian mappings*, Proceedings of the 10th International Conference on Fixed Point Theory and Its Applications, July 9-18, 2012, Cluj-Napoca, Romania, pages 209-216, House of the Book of Science 2013.

[P2014] Ł. Piasecki, *Renormings of c_0 and the minimal displacement problem*, Ann. Univ. Mariae Curie-Skłodowska, Sec. A, 68 (2) (2014), 85-91.

[GP2014] K. Goebel and Ł. Piasecki, *Minimal displacement and related problems, revisited*, Proceedings of the Fourth International Symposium on Banach and Function Spaces 2012, Kitakyushu, Japan, September 12-15, 2012, Yokohama Publishers, 2014.

[PP2016] V. Pérez-García and Ł. Piasecki, *From mean lipschitzian mappings to a generalized moving averages in Banach spaces*, J. Nonlinear Convex Anal. 17 (3) (2016), 589-597.

[CP2017] E. Casini and Ł. Piasecki, *The minimal displacement and optimal retraction problems in some Banach spaces spaces*, J. Nonlinear Convex Anal. 18 (1) (2017), 61-71.

[PPS2017] V. Pérez-García, Ł. Piasecki, A. Sánchez Nungaray, *Solving polynomials with Fibonacci type sequences*, J. Nonlinear Convex Anal. 18 (2) (2017), 251-259.

8.1. **The optimal retraction and minimal displacement problems.** In 1912, L. E. J. Brouwer [16] proved that every bounded, closed and convex set C in a finite-dimensional Banach space X has the *topological fixed point property* i.e. every continuous mapping $T : C \to C$ has a fixed point. It is well known that this result has an equivalent form saying that the unit sphere S_X in a finite-dimensional Banach space X is *not a retract* of the closed unit ball B_X i.e. there is no continuous map $R : B_X \to S_X$ (*a retraction*) such that $Rx = x$ for all $x \in S_X$. The most popular and useful result extending Brouwer's Fixed Point Theorem to the case of infinite-dimensional Banach spaces was formulated in 1930 by J. P. Schauder [73] and states that every convex, compact subset C of a Banach space X has the topological fixed point property. Already at this very stage, a natural question arises. What happens if C is noncompact? A full answer to this question came in 1953 and 1955 V. Klee's papers [45], [46]. Indeed, he proved that for every noncompact, closed, and convex subset C of a Banach space X, there exists a continuous, fixed point free mapping $T : C \to C$. In view of Schauder's and Klee's results, we conclude that a closed and convex set C in a Banach space X has the topological fixed point property if and only if it is compact.

Another approach to this issue was initiated by Kazimierz Goebel in his work from 1973 [27], in which he gave examples of lipschitzian mappings $T : C \to C$ with the *minimal displacement*

$$d(T) = \inf \{\|x - Tx\| : x \in C\} > 0.$$

Further research led to the formulation of theorems much stronger than Klee's result. Indeed, in 1979, Nowak [65] proved that for a certain class of Banach spaces X the sphere is a *lipschitzian retract* of the ball, that is, there exists a retraction $R : B_X \to S_X$ satisfying with a certain constant k the following condition: $\|Rx - Ry\| \leq k \|x - y\|$ for all $x, y \in B_X$. Then, four years later, Benyamini and Sternfeld [7] proved that this is true for any infinite dimensional Banach space X. The strongest result in this matter was obtained in 1985 by Lin and Sternfeld [56]. Their result states that for every bounded, closed, convex but noncompact set C in a Banach space X and for every $k > 1$, there exists a k-lipschitzian mapping $T : C \to C$ (i.e. $\|Tx - Ty\| \leq k \|x - y\|$ for all $x, y \in C$) with $d(T) > 0$. In particular, this situation occurs in the special case of the closed unit ball B_X of any infinite-dimensional Banach space X i.e. there exists k-lipschitzian mapping $T : B_X \to B_X$ with $d(T) > 0$, and every such mapping can be applied to construct a lipschitzian retraction R of B_X onto S_X (for more details see for e.g. [28]).

Current research in this area is focused on two basic problems: the *optimal retraction problem* and the *minimal displacement problem.*

The optimal retraction problem deals with the following issue: for a given Banach space X, find the exact value or a good estimate of the *optimal retraction constant* defined by

$$k_0(X) = \inf \{k : \text{there exists } k\text{-lipschitzian retraction } R \text{ of } B_X \text{ onto } S_X\}.$$

At present, the exact value of $k_0(X)$ is not known for any single Banach space X. We will now present the best of the currently known estimates.

We begin with the first basic estimate from below stating that $k_0(X) \geq 3$ for every Banach space X (see [31]). However, for some particular spaces better estimates are known. For example, K. Bolibok [12] proved that $k_0(\ell_1) \geq 4$ and in [CP2017] it is shown that for a Hilbert space H we have $k_0(H) > 4.58$. Much more efforts were devoted to give a reasonable estimate from above. In 2007, M. Annoni and E. Casini [4] proved that $k_0(\ell_1) \leq 8$; the previous estimate was $k_0(\ell_1) < 9.43$ (see [28]). Soon after, the same estimate was obtained for the space $L_1(0,1)$ by K. Goebel, G. Marino, L. Muglia and R. Volpe [34]. Yet another simple construction showing that $k_0(L_1(0,1)) \leq 8$ is presented in Example 9.17 in [P2013]. M. Baronti, E. Casini and C. Franchetti [8] proved that for a Hilbert space H we have $k_0(H) \leq 28.99$; the previous estimates were: $k_0(H) \leq 64.25$ by T. Komorowski and J. Wośko [48], and $k_0(H) \leq 31.45$ by K. Bolibok [12].

In my master's thesis [69], written under the guidance of Professor Kazimierz Goebel, I showed that for the spaces c_0, c, $C[0,1]$ and $BC(\mathbb{R})$, we have $k_0(X) \leq 4(2 + \sqrt{3}) = 14.92\ldots$; the previous estimate for these spaces was $k_0(X) \leq 4(1 + \sqrt{2})^2 = 23.31\ldots$ (see

[28] and [34]). Moreover, in [69], I proved that $k_0(BC_z(\mathrm{M})) \leq 2(2+\sqrt{2}) = 6.828\ldots$, where (M,d) is a connected metric space consisting of more than one point, $z \in M$ is a given point, and $BC_z(M)$ denotes the space of all bounded, continuous functions $f : M \to \mathbb{R}$ vanishing at z, $f(z) = 0$, and equipped with the standard sup norm. (The previous estimates for this space were, consecutively, $k_0(C_0([0,1])) \leq 15.82$ in [12], $k_0(BC_z(\mathrm{M})) \leq 12$ in [33] and $k_0(BC_z(\mathrm{M})) \leq 7$ in [GP2008]). So far, the above estimates are the best known. Moreover, the estimate $k_0(BC_z(\mathrm{M})) \leq 2(2+\sqrt{2}) = 6.828\ldots$ is the minimum of upper bounds over all the Banach spaces for which the upper bound is known. My master's thesis won the award of the Polish Mathematical Society Competition in honour of Józef Marcinkiewicz (1910-1940) for the best student thesis on any branch of mathematics (second prize). The results contained therein have been published in *Topological Methods in Nonlinear Analysis* and *Nonlinear Analysis* (see [P2009] and [P2011]).

Very recently, it was proved that $k_0(l_\infty) \leq 12+2\sqrt{30} = 22.95\cdots$, and a general estimate for some subspaces of spaces of continuous functions was given (see [CP2017]).

The optimal retraction problem is closely related to another nontrivial problem posed by Goebel in 1973. Suppose that C is a bounded, closed, convex and noncompact subset of a Banach space X. The *minimal displacement* of a mapping $T : C \to C$ is the number

$$d(T) = \inf\left\{\|x - Tx\| : x \in C\right\}.$$

Moreover, the function $\varphi_C : [1,+\infty) \to [0, \operatorname{diam}(C)]$ defined by

$$\varphi_C(k) = \sup\left\{d(T) : T : C \to C,\ T \text{ is } k\text{-lipschitzian}\right\}$$

is called the *characteristic of minimal displacement of* C. In the special case, when $C = B_X$, we write ψ_X instead of φ_X.

It is known (see [31]) that for any C as above and for every $k \geq 1$, $\varphi_C(k) \leq \left(1 - \frac{1}{k}\right) r(C)$, where $r(C)$ denotes the Chebyshev radius of C, i.e. $r(C) = \inf_{z\in C} \sup\left\{\|z-y\| : y \in C\right\}$. A set C is called *extremal* if $\varphi_C(k) = \left(1-\frac{1}{k}\right) r(C)$ for every $k \geq 1$. A space X is named an *extremal space* if its unit ball B_X is extremal.

The characteristic of minimal displacement can be considered not only for the whole class of lipschitzian mappings but also for its various subclasses. Among them, the most interesting are:

$$\psi_{B_X \to S_X}(k) = \sup\left\{d(T) : T : B_X \to S_X,\ T \text{ is } k\text{-lipschitzian}\right\}$$

and

$$\psi_{S_X \to \{0\}}(k) = \sup\left\{d(T) : T : B_X \to B_X,\ T \text{ is } k\text{-lipschitzian and } T(S_X) = \{0\}\right\}.$$

The minimal displacement problem deals with finding or evaluating the functions mentioned above, for concrete sets or spaces. The exact values of these functions are only known for extremal sets and spaces. Among them are: $C[0,1]$, $C_0[0,1]$, c_0, the positive face S^+ in ℓ_1 (see [28]).

In [69] (see also [P2011]), it is proved that the space c is also extremal.

In [GP2014], it is proved that for the space $X = C_0[0,1]$ we have

$$\left(1-\frac{1}{k}\right)\min\left\{\frac{k+1}{4}, 1\right\} \leq \psi_{S_X\to\{0\}}(k) \leq \left(1-\frac{1}{k}\right)\min\left\{\frac{k}{2}, 1\right\}.$$

In particular, $\psi_{S_X\to\{0\}}(k) = 1 - \frac{1}{k}$ for all $k \geq 3$.

The space ℓ_1 is not extremal and we have $\psi_X(k) < \varphi_{\frac{1}{2}S^+}(k) = 1 - \frac{1}{k}$ for every $k > 1$ (see [28]). A Hilbert space H is also not extremal but in this case for any set $C \subset H$ with $r(C) = 1$ we have $\varphi_C(k) = \psi_H(k)$ for all $k \geq 1$ (see [28]). In particular, Goebel [27] proved that

$$\psi_H(k) \leq \left(1 - \frac{1}{k}\right)\sqrt{\frac{k}{k+1}}$$

and Casini [20] showed that

$$\psi_H(k) \geq 1 - \frac{2\sqrt{\sqrt{2}(k+1)}}{k}.$$

Moreover, in 1973 Goebel [27] proved that

$$\psi_{B_H \to S_H}(k) \leq \left(1 - \frac{1}{k}\right)^{3/2}. \tag{8.1}$$

Very recently this estimate was improved in [CP2017], where the following upper bound is given:

$$\psi_{B_H \to S_H}(k) \leq \left(1 - \frac{1}{k}\right)^{3/2}\left(\frac{2k+1}{2}\right)\sqrt{\frac{k-1}{k^3 + k - \sqrt{k(3k+1)}}}.$$

Recall that c_0 is extremal. Therefore, it is straightforward to see that if X contains almost isometric copies of c_0, then for every $\varepsilon > 0$ there exists a bounded closed and convex set C in X such that $\varphi_C(k) > 1 - \frac{1}{k} - \varepsilon$ for all $k > 1$. A much more subtle result can be found in [P2014], where it is proved that if a Banach space X contains an *asymptotically isometric* copy of c_0, then X contains a bounded closed and convex set C with $r(C) = 1$ such that for every $k \geq 1$ there exists a k-contractive mapping $T : C \to C$ (i.e. $\|Tx - Ty\| < k\,\|x - y\|$ for all $x, y \in C$, $x \neq y$) satisfying $\|Tx - x\| > 1 - \frac{1}{k}$ for all $x \in C$.

8.2. **Classification of Lipschitz Mappings.** In 2007, Kazimierz Goebel and Maria Japón Pineda [29] introduced a class of *mean nonexpansive mappings* and proved some fixed point theorems for them. Then, this definition was extended in [35]: let (M, ρ) be a metric space and $T : M \to M$, suppose that $\alpha = (\alpha_1, \ldots, \alpha_n)$, where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, $\alpha_i \geq 0, \alpha_1 > 0, \alpha_n > 0$ and $\sum_{j=1}^{n} \alpha_j = 1$; we say that T is an α-lipschitzian mapping for the constant k if for each $x, y \in M$

$$\sum_{j=1}^{n} \alpha_j \rho(T^j x, T^j y) \leq k\rho(x, y).$$

When the multi-index α and the constant k are not specified, we simply say that T is mean lipschitzian. If $k = 1$, then we say that T is mean nonexpansive.

The above condition was in my area of interest. Since, at that time, Victor Pérez-García held a post-doc position at Maria Curie-Skłodowska University, I decided to invite him to a joint project. Below I will briefly recall some main results of our fruitful collaboration.

The mean Lipschitz condition involves only a finite number of iterates. In spite of this, it turned out that this condition has a serious influence not only on the behavior of the sequence of Lipschitz constants for consecutive iterates T^n of T but also on its asymptotic behavior (see [PP2011], [PP2012], [PP2013] and [PP2016]).

In [PP2011], we gave a sharp evaluation of the Lipschitz constants for iterates of mean lipschitzian mappings:

Theorem 8.1 (Theorem 2.1 in [PP2011])**.** *Let (M, ρ) be a metric space, suppose that $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_1 > 0$, $\alpha_i \geq 0$ and $\sum_{j=1}^{n} \alpha_j = 1$, $n \geq 1$, and let $T : M \to M$ be* *an α-lipschitzian mapping with the constant k. Then $k(T^m) \le b_m$, where the sequence $\{b_m\}_{m=0}^{\infty}$ is defined as follows:*

$$b_m = \begin{cases} 1 & \text{for} \quad m = 0, \\ \dfrac{k}{\sum_{j=1}^{m} \alpha_j b_{m-j}^{-1}} & \text{for} \quad m = 1, \cdots, n, \\ \dfrac{k}{\sum_{i=1}^{n} \alpha_i b_{m-i}^{-1}} & \text{for} \quad m = n+1, n+2, \cdots . \end{cases}$$

In the same paper we proved that this bound is sharp ([PP2011], Example 2.2). We used the mapping $T : \ell_1 \to \ell_1$ defined for every $x = (x_1, x_2, \dots) \in \ell_1$ by

$$Tx = \left(\frac{b_1}{b_0} x_2, \frac{b_2}{b_1} x_3, \dots, \frac{b_j}{b_{j-1}} x_{j+1}, \dots \right). \tag{$\heartsuit$}$$

We also proved similar results for the class of (α, p)-lipschitzian mappings with the constant k, that is, $\sum_{j=1}^{n} \alpha_j \rho(T^j x, T^j y)^p \le k^p \rho(x, y)^p$ (see Theorem 2.3 in [PP2011]).

In [PP2012], we studied the asymptotic behavior of the sequence of Lipschitz constants for iterates of mean nonexpansive mappings, that is, for $k = 1$. We defined $d_m = 1/b_m$ to obtain the relation:

$$d_m = \begin{cases} 1 & \text{for} \quad m = 0, \\ \sum_{j=1}^{m} \alpha_j d_{m-j} & \text{for} \quad m = 1, \cdots, n, \\ \sum_{i=1}^{n} \alpha_i d_{m-i} & \text{for} \quad m = n+1, n+2, \cdots . \end{cases}$$

We proved one result concerning localization of roots of polynomials (Lemma 2.4. in [PP2012]). We used this to show that the eigenvalues of a special matrix lie strictly inside the complex unit disc. Using also a special property of the mapping T (see Lemma 2.1 in [PP2012]), we finally obtained (see Theorem 2.5 in [PP2012])

$$\lim_{m \to \infty} d_m = \frac{1}{\sum_{j=1}^{n} \left(\sum_{i=j}^{n} \alpha_i \right)}$$

and so

$$\lim_{m \to \infty} b_m = \sum_{j=1}^{n} \left(\sum_{i=j}^{n} \alpha_i \right).$$

Consequently, we obtained the following

Theorem 8.2 (Theorem 2.6 in [PP2012]). *If $T : M \to M$ is (α, p)-nonexpansive, then*

$$\limsup_{m \to \infty} k(T^m) \le \left(\lim_{m \to \infty} b_m \right)^{\frac{1}{p}} = \left(\sum_{j=1}^{n} \left(\sum_{i=j}^{n} \alpha_i \right) \right)^{\frac{1}{p}}.$$

Furthermore, we applied this result to obtain some new fixed point theorems for mean nonexpansive mappings (see Theorem 3.4 and Corollaries 3.5-3.7 in [PP2012]). Perhaps the most interesting is the following

Corollary 8.3 (Corollary 3.7 in [PP2012]). *Let C be a nonempty, convex, closed and bounded subset of a Hilbert space H and $\alpha = (\alpha_1, \alpha_2)$. Then C has the fixed point property for all $(\alpha, 2)$-nonexpansive mappings.*

Let $T : M \to M$ be a lipschitzian mapping. The formula

$$k_\infty(T) = \lim_{m\to\infty} \left(k(T^m)\right)^{1/m},$$

defines the so-called *spectral radius* of T, which in the case of nonlinear mappings has the following interpretation:

$$k_\infty(T) = \inf\left\{k_d(T) : d \text{ is equivalent to the metric } \rho\right\},$$

where $k_d(T)$ means the Lipschitz constant for T with respect to the metric d.

In [PP2013], given $\alpha = (\alpha_1, \dots, \alpha_n)$ as above and $k > 0$, we gave a sharp bound for $k_\infty(T)$, where T is any α-lipschitzian mapping with the constant k.

Theorem 8.4 (Theorem 2.1 in [PP2013])**.** *Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be as our general assumption, $k > 0$ and $\{b_m\}_{m=0}^\infty$ as defined in Theorem 8.1. Let g be the unique positive solution of the equation*

$$\alpha_1 t + \alpha_2 t^2 + \dots + \alpha_n t^n = k,$$

then $\lim_{m\to\infty} (b_m)^{1/m} = g$.

It is worth mentioning that the main trick in the proof of the above theorem was to define a new norm in ℓ_1, given for every $x \in \ell_1$ by

$$\begin{aligned} \|x\|_T = &\left(\alpha_1 + \alpha_2 g + \dots + \alpha_n g^{n-1}\right)\|x\| \\ &+ \left(\alpha_2 + \alpha_3 g + \dots + \alpha_n g^{n-2}\right)\|Tx\| \\ &+ \left(\alpha_3 + \alpha_4 g + \dots + \alpha_n g^{n-3}\right)\|T^2x\| + \dots \\ &+ \alpha_n\|T^{n-1}x\|, \end{aligned}$$

where T is the mapping given by ($\heartsuit$).

From Theorem 8.4 and Theorem 8.2 we obtained the following

Corollary 8.5. *Let (M, ρ) be a metric space and $T : M \to M$ an α-lipschitzian mapping with $\alpha = (\alpha_1, \dots, \alpha_n)$ and $k > 0$. Then*

$$k_\infty(T) \le g,$$

where g is a unique positive solution of the equation

$$\alpha_1 t + \alpha_2 t^2 + \dots + \alpha_n t^n = k.$$

By following the tricks used in [PP2012], we can easily solve the following *moving average problem*: find the limit of the sequence $\{d_m\}_{m=0}^\infty$, where $d_0, \dots, d_{n-1}$ are arbitrary numbers and

$$d_m = \alpha_1 d_{m-1} + \dots + \alpha_n d_{m-n} \quad \text{for } m \ge n.$$

In [PP2016], we solved a more general problem for the so-called *generalized moving averages* (or *Fibonacci type sequences*) in the general case of Banach spaces:

Theorem 8.6 (Theorem 3.3 in [PP2016])**.** *Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be as before, $k > 0$, and g be the unique positive solution of the equation*

$$\alpha_1 t + \alpha_2 t^2 + \dots + \alpha_n t^n = k.$$

Let $x_0, \dots, x_{n-1}$ be arbitrary elements of a Banach space X and x_m be defined for $m \ge n$ by

$$x_m = \sum_{i=1}^n \frac{\alpha_i}{k} x_{m-i}.$$

Then

$$\lim_{m\to\infty} g^m x_m = \frac{1}{\sum_{i=1}^n \sum_{j=i}^n \alpha_j g^j} \sum_{i=1}^n \left(\sum_{j=i}^n \alpha_j g^j \right) g^{n-i} x_{n-i}.$$

In particular, if $k = 1$, then we obtain the classical moving averages and the solution of the problem mentioned before:

$$\lim_{m\to\infty} x_m = \frac{1}{\sum_{i=1}^n \sum_{j=i}^n \alpha_j} \sum_{i=1}^n \left(\sum_{j=i}^n \alpha_j \right) x_{n-i}.$$

In [PPS2017], by applying the techniques developed in [PP2012], [PP2013] and [PP2016], we gave a new algorithm which is useful to find the unique positive root of a certain class of polynomials.

Many results presented above constituted an important part of my PhD thesis [70], written under the guidance of Professor Kazimierz Goebel. Its extended version has been published in 2013 as a book "Classification of Lipschitz Mappings" CRC Press (see [P2013]).

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