JOANNA JURASIK

Abstract of the doctoral thesis

Properties of asymmetric truncated Toeplitz operators

Let $\mathbb{D}$ denote the unit disc, and let $\mathbb{T}$ denote the unit circle. By $L^2 := L^2(\mathbb{T}, \frac{d\theta}{2\pi})$ we will be denote the space of all Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{C}$ such that

$$\|f\|_{L^2} = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta \right)^{\frac{1}{2}} < \infty.$$

H^2 is the usual Hardy space, the subspace of L^2 of normalized Lebesgue measure on $\mathbb{T}$ whose negative indexed Fourier coefficients are all zero. H^2 will interchangably refer to both the boundary functions and the functions on $\mathbb{D}$. Let P denote the projection from L^2 to H^2. Let S denote the shift operator $f \mapsto zf$ on H^2. Its adjoint (the backwards shift) is the operator

$$(S^*f)(z) = \frac{f(z) - f(0)}{z}.$$

A Toeplitz operator is the compression of a multiplication operator on L^2 to H^2. In other words, given $\varphi \in L^2$ (called the symbol of the operator),

$$T_\varphi f = P(\varphi f)$$

is the operator that sends f to $P(\varphi f)$ for all $f \in H^2$. This operator is bounded if and only if $\varphi \in L^\infty$ (the space of all essentially bounded measurable functions on $\mathbb{T}$).

Chapter 1 of this dissertation has an introductory character. Moreover, we present basic properties of Hardy spaces and Toeplitz operators. In [2] A. Brown and P. R. Halmos describe the algebraic properties of Toeplitz operators. Among other things, they found necessary and sufficient conditions for a bounded operator on H^2 be a Toeplitz operator, namely a bounded operator $T : H^2 \to H^2$ is a Toeplitz operator if and only if $S^*TS = T$.

Let H^∞ be the algebra of bounded analytic functions on $\mathbb{D}$ and let $\alpha \in H^\infty$ be an arbitrary inner function, that is, $|\alpha| = 1$ a.e. on $\mathbb{T}$. By the theorem of A. Beurling (see, for example, [10, Thm. 17.21]), every nontrivial, closed S-invariant subspace of H^2 can be expressed as αH^2 for some inner function α. Consequently, every nontrivial, closed S^*-invariant subspace of H^2 is of the form

$$K_\alpha = H^2 \ominus \alpha H^2$$

with α inner. The space K_α is called the model space corresponding to α.

In Chapter 2 we deal with the so-called truncated Toeplitz operators. Let P_α denote the orthogonal projection of L^2 onto K_α. Truncated Toeplitz operators are operators A_φ^α, $\varphi \in L^2$, densly defined on the model spaces K_α, by the formula

$$A_\varphi^\alpha f = P_\alpha(\varphi f).$$

The operator A_φ^α can be seen as a compression to K_α of the classical Toeplitz operator T_φ.

The study of truncated Toeplitz operators as a class began in 2007 with D. Sarason's paper [11]. In spite of similar definitions (for example, $(A_\varphi^\alpha)^* = A_{\overline{\varphi}}^\alpha$), there are many differences between truncated Toeplitz operators and the classical ones. One of the first results from [11] states that, unlike in the classical case, a truncated Toeplitz operator is not uniquely determined by its symbol. More precisely, $A_\varphi^\alpha = 0$ if and only if $\varphi \in \overline{\alpha H^2} + \alpha H^2$ ([11, Thm. 3.1]). Moreover, unlike in the classical case, unbounded symbols can produce bounded truncated Toeplitz operators and there are bounded truncated Toeplitz operators for which no bounded symbol exists (see [1]).

The compression of S to K_α will be denoted by S_α. Its adjoint, S_α^*, is the restriction of S^* to K_α. The operators S_α and S_α^* are the truncated Toeplitz operators with symbols z and $\overline{z}$, respectively. The bounded operator $A : K_\alpha \to K_\alpha$ is a truncated Toeplitz operator if and only if there are functions $\chi, \psi \in K_\alpha$ such that

$$A - S_\alpha^* A S_\alpha = \psi \otimes \widetilde{k}_0^\alpha + \widetilde{k}_0^\alpha \otimes \chi,$$

where $\widetilde{k}_0^\alpha(z) = \frac{\alpha(z)-\alpha(0)}{z}$ ($\otimes$ is rank one operator on Hilbert space, $f \otimes g(h) = \langle h, g \rangle f$, for f, g and h from this space) (see [11, 4.1]). More background about model spaces and truncated Toeplitz operators can be found in Chapter 2.

If α has distinct zeros $\{a_1, \ldots, a_m\}$ and

$$k_w^\alpha(z) = \frac{1 - \overline{\alpha(w)}\alpha(z)}{1 - \overline{w}z}, \qquad \widetilde{k}_w^\alpha(z) = \frac{\alpha(z) - \alpha(w)}{z - w}, \ w \in \mathbb{D}$$

then the set $\mathcal{R}_m^\alpha = \{k_{a_1}^\alpha, \ldots, k_{a_m}^\alpha\}$ as well as $\widetilde{\mathcal{R}}_m^\alpha = \{\widetilde{k}_{a_1}^\alpha, \ldots, \widetilde{k}_{a_m}^\alpha\}$ is a (non-orthonormal) basis for K_α. In 2008 [6] J.A. Cima, W.T. Ross and W.R. Wogen considered truncated Toeplitz operators on finite-dimensional model spaces. The authors in [6] characterized truncated Toeplitz operators in terms of the matrix representations with respect to each of these bases. They showed that a matrix representing a truncated Toeplitz operator on m-dimensional model space is completely determined by $2m - 1$ of its entries, those along the main diagonal and the first row (and the first row can be replaced by any other row or column). They also proved a similar result for the so-called Clark bases. Matrix representations of truncated Toeplitz operators on infinite-dimensional model spaces were considered in [9].

Recently, the authors in [3, 4, 5] introduced a generalization of truncated Toeplitz operators, the so-called asymmetric truncated Toeplitz operators. Let α, β be two inner functions and let $\varphi \in L^2$. An asymmetric truncated Toeplitz operator $A_\varphi^{\alpha,\beta}$ is the operator from K_α into K_β given by

$$A_\varphi^{\alpha,\beta} f = P_\beta(\varphi f), \quad f \in K_\alpha.$$

The operator $A_\varphi^{\alpha,\beta}$ is densely defined. Clearly, $A_\varphi^{\alpha,\alpha} = A_\varphi^\alpha$. Let

$$\mathscr{T}(\alpha, \beta) = \{A_\varphi^{\alpha,\beta} \ : \ \varphi \in L^2(\partial\mathbb{D}) \text{ and } A_\varphi^{\alpha,\beta} \text{ is bounded}\}$$

and $\mathscr{T}(\alpha) = \mathscr{T}(\alpha, \alpha)$.

Chapter 3 describes properties of so-called asymmetric truncated Toeplitz operators. We describe when an operator from $\mathcal{T}(\alpha, \beta)$ is equal to the zero operator. The description is given in terms of the symbol of the operator. This was done in [3] and [4] for the case when β divides α, that is, when α/β is an inner function. It was proved in [3] and [4] that $A_\varphi^{\alpha,\beta}=0$ if and only if $\varphi\in\overline{\alpha H^2}+\beta H^2$. Here we show that this is true for all inner functions α and β.

We note that if α is a finite Blaschke product of degree m, then K_α has dimension m. By elementary linear algebra, the complex vector space of all linear transformations on K_α has dimension m^2. D. Sarason [11, Thm. 3.1] proved that if α is a finite Blaschke product of degree $m>0$, then the dimension of $\mathscr{T}(\alpha)$ is $2m-1$. We show that if α and β are finite Blachke products of degree $m>0$ and $n>0$, respectively, then the dimension of $\mathscr{T}(\alpha,\beta)$ is $m+n-1$. We also give some examples of rank-one asymmetric truncated Toeplitz operators.

In chapter 4 we generalize the results from [6] concerning matrix representations. We characterize matrix representations of asymmetric truncated Toeplitz operators acting between finite-dimensional model spaces. We prove theorem

Theorem. *Let the function α be a finite Blaschke product with m distinct zeros $a_1,\dots,a_m$, let β be a finite Blaschke product with n distinct zeros $b_1,\dots,b_n$ and assume that α and β have precisely l zeros in common: $a_i=b_i$ for $i\le l$ ($l=0$ if there are no zeros in common). Let A be any linear transformation from K_α into K_β. If $M_A=(r_{s,p})$ is the matrix representation of A with respect to the bases $\mathcal{R}_m^\alpha=\{k_{a_1}^\alpha,\dots,k_{a_m}^\alpha\}$ and $\mathcal{R}_n^\beta=\{k_{b_1}^\beta,\dots,k_{b_n}^\beta\}$, and*

(a) *$l=0$, then $A\in\mathscr{T}(\alpha,\beta)$ if and only if*

$$r_{s,p}=\frac{\overline{\beta'(b_s)}(\bar a_1-\bar b_s)r_{s,1}+\overline{\beta'(b_1)}(\bar b_1-\bar a_1)r_{1,1}+\overline{\beta'(b_1)}(\bar a_p-\bar b_1)r_{1,p}}{\overline{\beta'(b_s)}(\bar a_p-\bar b_s)}\tag{1}$$

for all $1\le p\le m$ and $1\le s\le n$;

(b) *$l>0$, then $A\in\mathscr{T}(\alpha,\beta)$ if and only if*

$$r_{s,p}=\frac{\overline{\beta'(b_1)}(\bar a_1-\bar b_s)r_{1,s}+\overline{\beta'(b_1)}(\bar a_p-\bar b_1)r_{1,p}}{\overline{\beta'(b_s)}(\bar a_p-\bar b_s)}\tag{2}$$

for all p,s such that $1\le p\le m$, $1\le s\le l$, $s\neq p$, and

$$r_{s,p}=\frac{\overline{\beta'(b_s)}(\bar a_1-\bar b_s)r_{s,1}+\overline{\beta'(b_1)}(\bar a_p-\bar b_1)r_{1,p}}{\overline{\beta'(b_s)}(\bar a_p-\bar b_s)}\tag{3}$$

for all p,s such that $1\le p\le m$, $l<s\le n$.

We also consider matrix representations with respect to bases: $\widetilde{\mathcal{R}}_m^\alpha$ and $\widetilde{\mathcal{R}}_n^\beta$, Clark bases $\mathcal{V}_m^\alpha$ and $\mathcal{V}_n^\beta$, modified Clark bases $\mathcal{E}_m^\alpha$ and $\mathcal{E}_n^\beta$, $\mathcal{R}_m^\alpha$ and $\widetilde{\mathcal{R}}_n^\beta$, $\widetilde{\mathcal{R}}_m^\alpha$ and $\mathcal{R}_n^\beta$, Clark base $\mathcal{V}_m^\alpha$ and $\mathcal{R}_n^\beta$, Clark base $\mathcal{V}_m^\alpha$ and $\widetilde{\mathcal{R}}_n^\beta$, $\widetilde{\mathcal{R}}_m^\alpha$ and Clark base $\mathcal{V}_n^\beta$, and base $\mathcal{R}_m^\alpha$ and Clark base $\mathcal{V}_n^\beta$.

We also characterize matrix representations of asymmetric truncated Toeplitz operators acting between infinite-dimensional model spaces.

References

[1] A. Baranov, I. Chalendar, E. Fricain, J. E. Mashreghi and D. Timotin, *Bounded symbols and reproducing kernel thesis for truncated Toeplitz operators,* J. Funct. Anal. 259 (2010), no. 10, 2673–2701.

[2] A. Brown, P. R. Halmos, *Algebraic properties of Toeplitz operators,* J. Reine Angew. Math. 213 (1963/1964), 89–102.

[3] C. Câmara, K. Kliś-Garlicka, J. Jurasik, M. Ptak, *Characterizations of asymmetric truncated Toeplitz operators,* Banach J. Math. Anal. 11 (2017), no. 4, 899âĂŞ922.

[4] M. C. Câmara, J. R. Partington, *Asymmetric truncated Toeplitz operators and Toeplitz operators with matrix symbol,* J. Oper. Theory 77 (2017), no. 2, 455–479.

[5] M. C. Câmara, J. R. Partington, *Spectral properties of truncated Toeplitz operators by eqvivalence after extension,* J. Math. Anal. Appl. 433 (2016), no. 2, 762–784.

[6] J. A. Cima, W. T. Ross, W. R. Wogen, *Truncated Toeplitz operators on finite dimensional spaces,* Oper. Matrices 2 (2008), no. 3, 357–369.

[7] J. Jurasik, B. Łanucha, *Asymmetric truncated Toeplitz operators equal to the zero operator,* Ann. Univ. Mariae Curie-SkÅĆodowska Sect. A 70 (2016), no. 2, 51–62.

[8] J. Jurasik, B. Łanucha, *Asymmetric truncated Toeplitz operators on finite-dimensional spaces,* Oper. Matrices 11 (2017), no. 1, 245–262.

[9] B. ÅĄanucha, *Matrix representations of truncated Toeplitz operators,* J. Math. Anal. Appl. 413 (2014), 430–437.

[10] W. Rudin, *Analiza Rzeczywista i Zespolona,* PWN, Warszawa 1998.

[11] D. Sarason, *Algebraic properties of truncated Toeplitz operators,* Operators and Matrices 1 (2007), no. 4, 491–526.

19.06. 2018r.

Joanna Jurasik