PhD thesis

"The density Turán problems for graphs and hypergraphs"

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Summary

The subject of the dissertation is the density Turán problem for graphs and hypergraphs. The name of this problem comes from Hungarian mathematician Pál Turán who formulated the following problem. For a given graph $G = (V(G), E(G))$ and a natural number n we want to determine a maximum number of edges (the Turán number $ex(G, n)$) in graph of order n which does not contain graph G as a subgraph. Each graph with this property is called an extreme graph of graph G.

Another Turán type problem considers a graph G together with some function determined on its set of edges $f : E(G) \to (0, 1]$ (this function associates to each edge some value called a density). This kind of problem seeks to determine some conditions, which the function f should satisfy to ensure the existence of a given graph G in each $|V(G)|$-partite graph $B[G]$ with edge densities set by the function f, where $B[G]$ is called *a blow-up graph* and is defined in the following way

1. first we repleace each vertex $i \in V(G)$ by a cluster A_i (a set of independent vertices).

2. next, if vertices $i, j \in V(G)$ are adjacent in G (that is an edge $e = ij \in E(G)$ exists) we create edges in $B[G]$ between vertices of clusters A_i and A_j (not necessarily all of them, but at least one).

A graph G is called *the transversal* in the blow-up graph $B[G]$ if there exists a homomorphism

$$\phi : V(G) \to V(B[G])$$

such that $\phi(i) \in A_i$ for all vertices $i \in V(G)$.

For a pair of clusters A_i and A_j in the blow-up graph $B[G]$, where $e = ij \in E(G)$, *an edge density*, denoted by $d_e = d(A_i, A_j)$, is defined as a sum of weights of edges between the clusters A_i and A_j.

In the density Turán problem we want to find a critical density (homogeneous case) or formulate an algoritm which tests whether the given function f ensures the existence of transversal G in each blow-up graph $B[G]$ with proper edge densities (inhomogeneous

case). In this dissertation we present known results of this subject matter for different classes of graphs, among others, for trees and cycles. The research was mostly inspired by papers [3], [4] and [5]. Furthermore, there are presented own results, which generalize the density Turán problem for other classes of graphs and hypergraphs (published in papers [1] and [2]).

For a graph G *the critical density* is defined as a minimum number d, which ensures the existence of transversal G in every blow-up graph of G with the property, that each edge denisty d_e satisfies the condition $d_e > d$.

In this work we extend the notion of critical density to hypergraph and present the lower and upper bounds for critical density for 3-uniform linear hypertrees. Those bounds are determined by the maximum eigenvalue of the adjacency matrix of hypertree. Furthermore, we give the hyperedge structure of optimal construction of extreme hypergraph for this kind of hypertrees.

Let $\{\gamma_e\}_G$ be a set of densities on edges in a graph G defined by the function f. We say that the set $\{\gamma_e\}_G$ ensures the existence of transversal G in each blow-up hypergraph $B[G]$ in which the condition $d_e \geq \gamma_e$ holds, if the graph G is a transversal in $B[G]$.

We state and prove theorems which describe necessary and sufficient conditions which the given set of densities must satisfy to ensure the existence of the transversal. Results included in this work concern unicyclic graphs with a cycle C_3, unihypercyclic linear hypergraph with a hypercycle C_3 and linear hypertrees. Additionally, we give and show correctness of algorithms which allow to test in efficient way if a given set of densities ensures, or does nor ensure, the existence of transversal for considered graphs and hypergraphs.

To prove obtained results we use methods of optimization, linear algebra, spectral graph theory and the notion of the multivariate matching polynomials of graphs and hypergraphs.

We state some open problems for further considerations.

Selected papers:

[1] H. BIELAK, K. POWROŹNIK, *An efficient algorithm for the density Turán problem of some unicyclic graphs*, Annals of Computer Science and Information Systems Proceedings of the 2014 FedCSIS 2 (2014) 479–486, http://dx.doi.org/10.15439/978-83-60810-58-3.

[2] H. BIELAK, K. POWROŹNIK, *The density Turán problem for some 3-uniform unihypercyclic linear hypergraphs. An efficient testing algorithm*, Annals of Computer Science and Information Systems Proceedings of the 2015 FedCSIS 5 (2015) 563–571, http://dx.doi.org/10.15439/2015F260.

[3] A. BONDY, J. SHEN, S. THOMASSÉ , C. THOMASSEN, *Density conditions for triangles in multipartite graphs*, Combinatorica 26 (2) (2006) 121-131.

[4] P. CSIKVÁRI, Z.L. NAGY, *The density Turán problem*, Comb. Probab. Comput. 21 (2012) 531-553.

[5] Z.L. NAGY, *A multipartite version of the Turán problem - density conditions and eigenvalues*, Electron. J. Comb. 18 (2011), #P46.

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