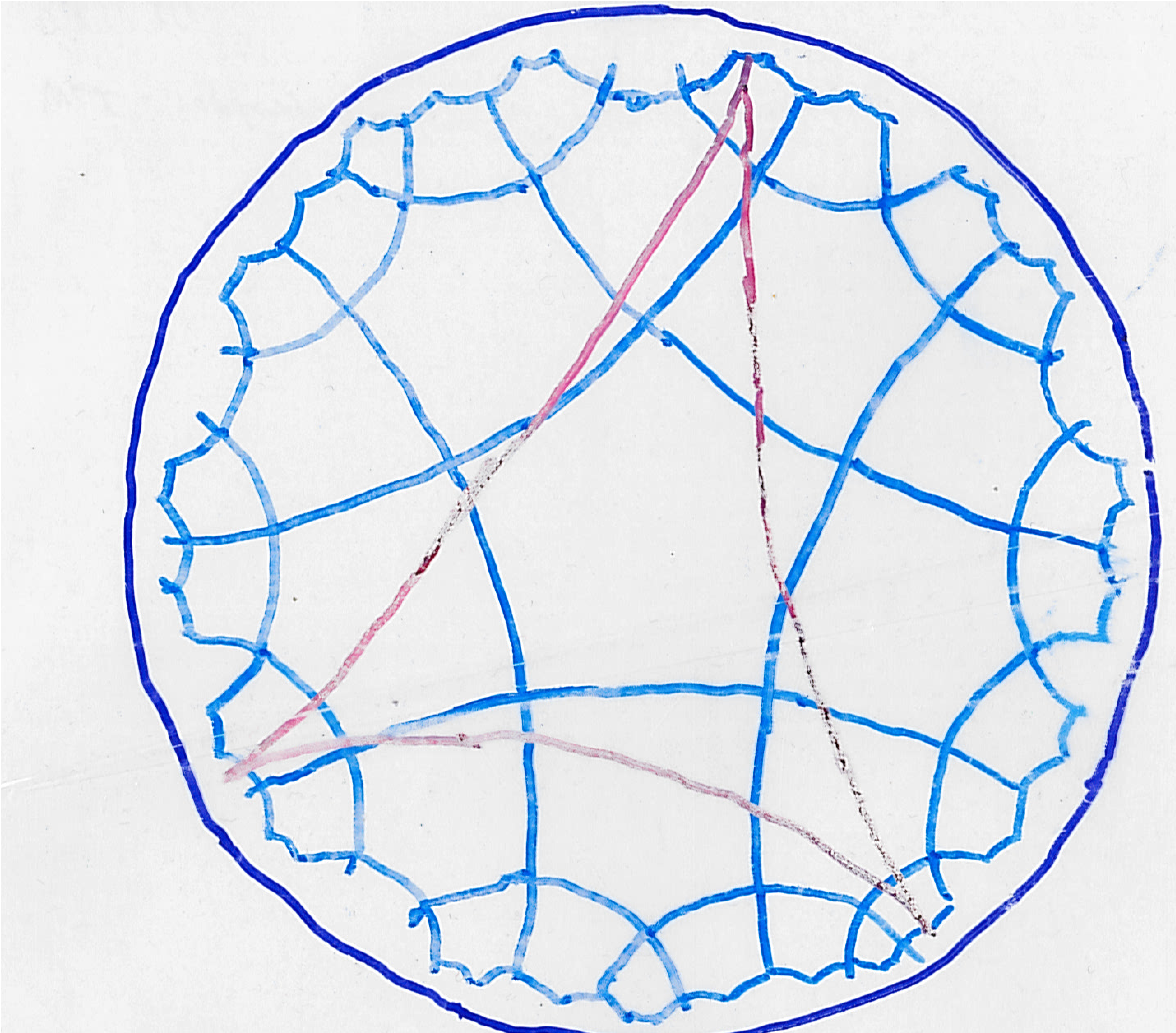
Porównywanie homologii singularnej z homologii Milnora Thurstona

[współautorzy: Janusz Przewocki (Gdańsk/Poznań) oraz Thilo Kuessner]



Porównywanie homologii singularnej z homologii Milnora Thurstona

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1.) The project:

[P(Dr)] Przewocki, Janusz: "Milnor-Thurston homology theory of some wild topological spaces", PhD-thesis, accepted at: Inst. Math. Polskej. Akad. Nauk 2015, available at: http://mat.ug.edu.pl/~jprzew/

[P(Publ)] Przewocki, Janusz: "Milnor-Thurston homology groups of the Warsaw Circle", Topology Appl. 160 (2013), no. 13, 1732-1741.

[PZ] Przewocki, Janusz; Zastrow, Andreas: "On the coincidence of zeroth Milnor-Thurston homology with singular homology", 2014, preprint submitted, available at: http://mat.ug.edu.pl/~jprzew/

[KPZ] Kuessner, Thilo; Przewocki, Janusz; Zastrow, Andreas: "On measure homology of mildly wild spaces", preprint

2.) Older work by the authors on that subject

[Kue] Kuessner, Thilo, “Efficient fundamental cycles of cusped hyperbolic manifolds”, Pacific J. Math. 211 (2003), no. 2, 283–313.

[Z(MTh)] Zastrow, Andreas: "On the (non)-coincidence of Milnor-Thurston homology theory with singular homology theory", Pacific J. Math., Vol. 186 (1998), no. 2, 369-396.

3.) Other quoted results

[BoZa] Bogopolski, Oleg; Zastrow, Andreas, "An uncountable homology group, where each element is an infinite product of commutators", preprint, submittted in revised version.

[FZ] Fischer, Hanspeter; Zastrow, Andreas: “Generalized universal covering spaces and the shape group. Fund. Math., Vol. 97 (2007), 167-196.”

[Grv] Gromov, Michael: “Volume and bounded cohomology”, Inst. Hautes Études Sci. Publ. Math. No. 56 (1982), 5–99 (1983).

[Iv] Ivanov, N. V., „Foundations of the theory of bounded cohomology”, Journal of Soviet Mathematics, Vol. 37, no. 3, 1090-1115

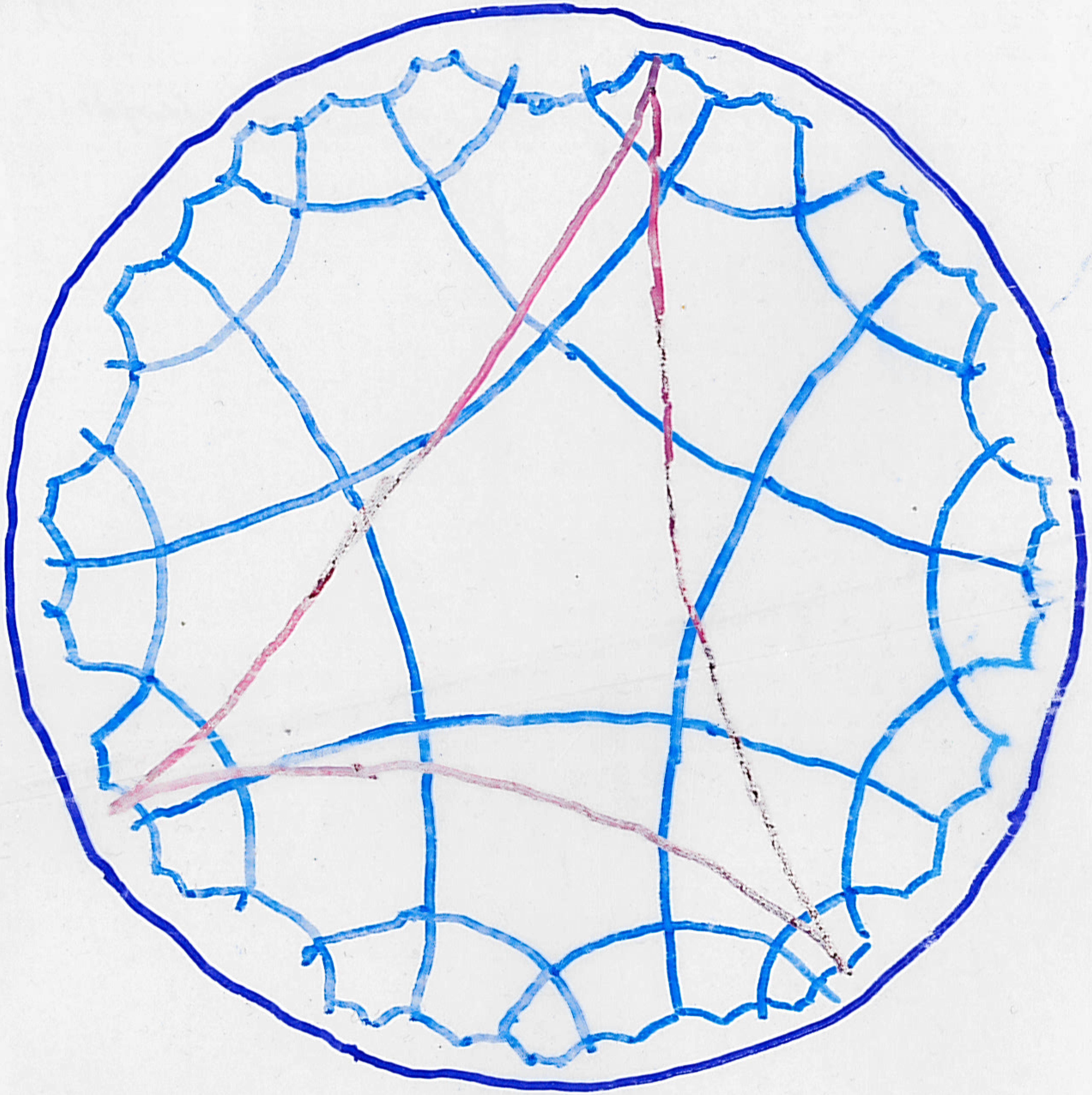
[Loeh] Löh, Clara: “Measure homology and singular homology are isometrically isomorphic”, Math. Z. 253 (2006), no. 1, 197–218.

[HM] Haagerup, Uffe & Munkholm, Hans J.: „Simplices of maximal volume in hyperbolic n-space”, Acta Math. 147 (1981), no. 1-2, 1-11.

[M-B] J.W. Milnor, M.G. Barratt: "An example of anamalous singular homology", Proc. AMS., Vol. 13 (1962), 293-297.

[Th] Thurston, W. P.: “Geometry and Topology of 3-manifolds”, mimographed lecture notes, Princeton, 1977

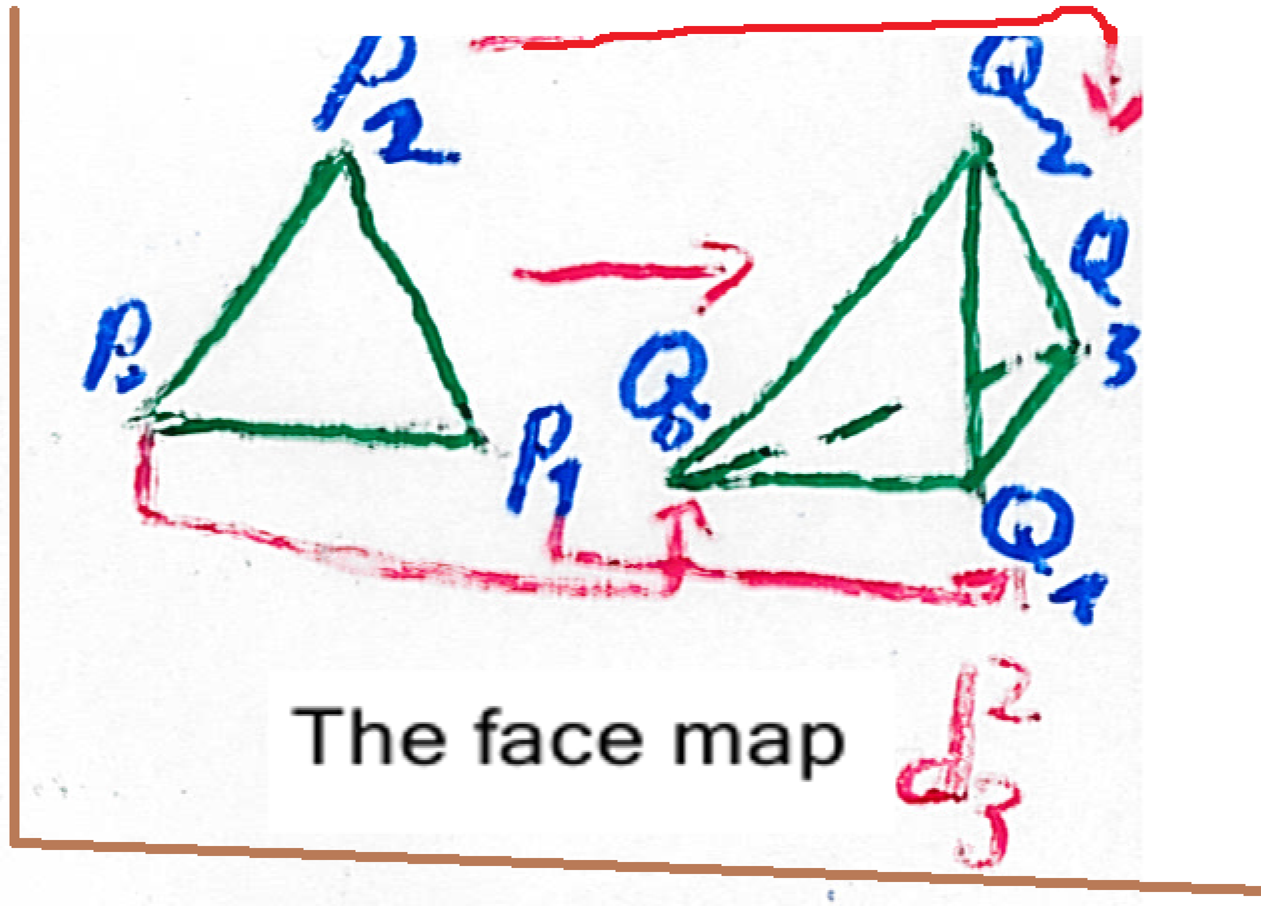
Original motivation of MTh-Homology Theory:



Already Thurston observed, that in order to avoid pathological cycles, one should restrict to considering measures

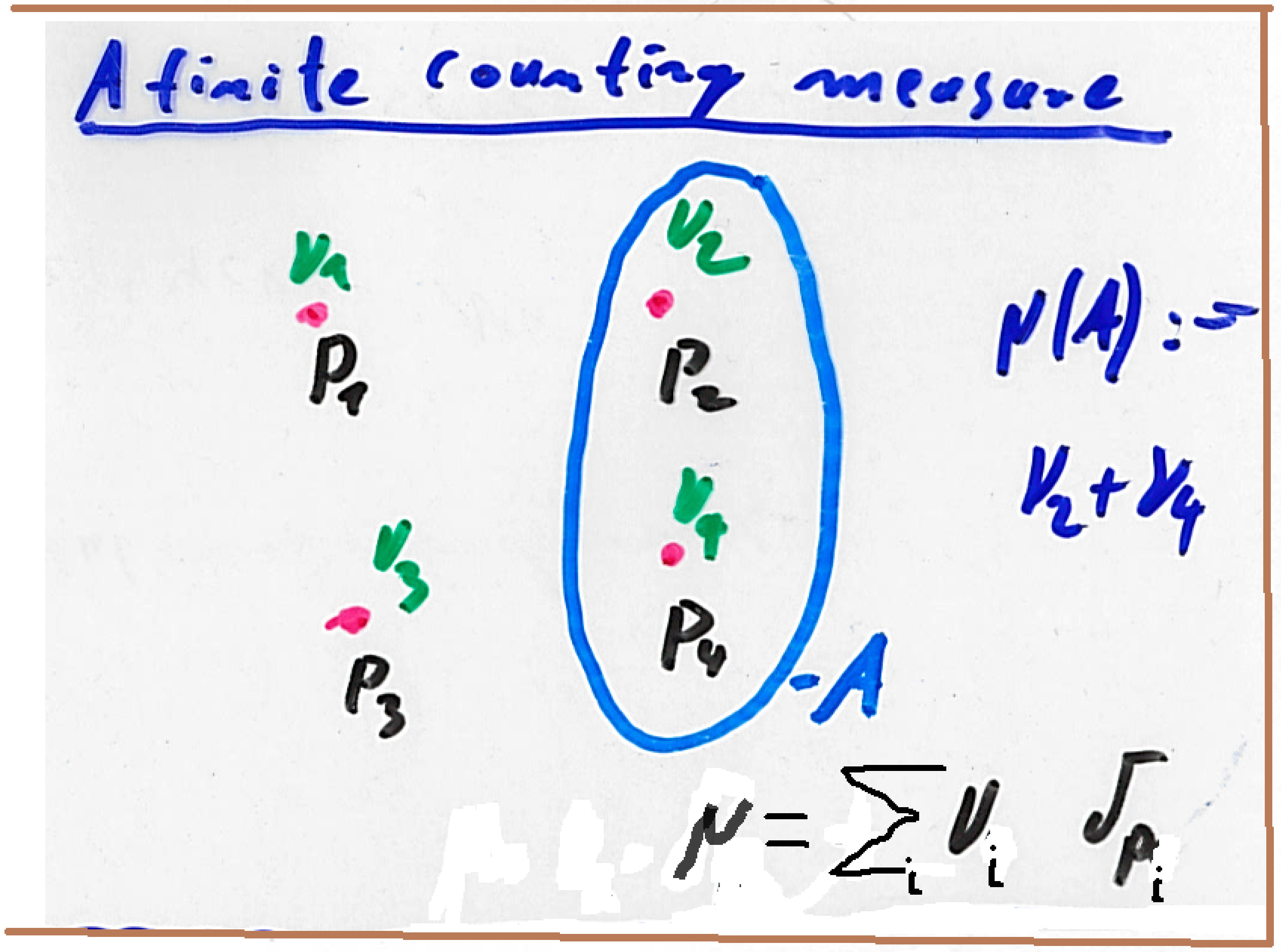
- that are bounded (in particular never $\mu(A) = \pm\infty$
- and that have compact carrier.
- It satisfies the Eilenerg-Steenrod Axioms (at least for normal spaces), and therefore gives the same homology groups as singular homology theory with real cofficients for all triangulable spaces.

Idea of Milnor-Thurston Homology Theory



- Replace classical singular chains $\sum_{i=1}^{n} a_i \sigma_i$ by measures on the set of all singular simplices
- a boundary of a measure can be constructed by observing:
  - face maps induce continuous maps $C(\Delta^k, X) \to C(\Delta^{k-1}, X)$
  - continuous maps $f: Y \to Z$ push measures forward according to $(f(\mu))(B) := \mu(f^{-1}(B))$, where $B \subseteq Z$

contains the measure μ



- Then $\partial_n(\mu) = \sum_{i=1}^{n} (-1)^i d_n^i(\mu)$ makes sense and defines the Milnor-Thurston-chain complex $\mathcal{C}_*(X)$

Define Milnor-Thurston Homology groups as homology-groups of the chain-complex

$$\mathcal{H}_k(X) := \mathcal{Z}_k(X) / \mathcal{B}_k(X)$$

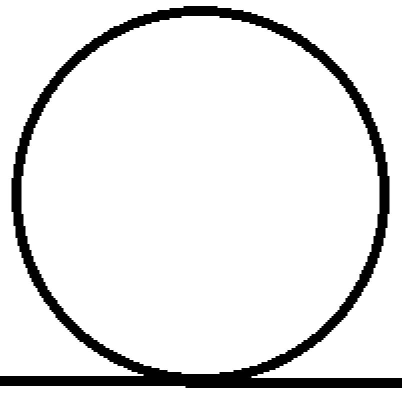
Associating $\sum_n a_i \sigma_i \mapsto$ corresponding counting measure induces canonical homomorphisms $H_k(X;\mathbb{R}) \to \mathcal{H}_k(X)$

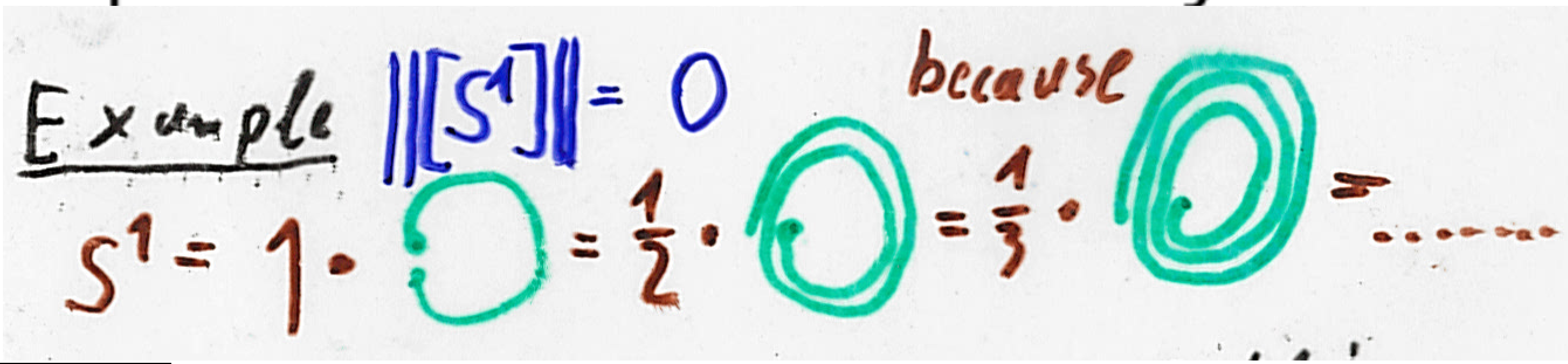
The definition of the Gromov-Norm (also called: the simplicial volume) extends from singular to Milnor-Thurston theory:

Ordinary Gromov-Norm:	Milnor-Thurston-Version of the Gromov Norm:
$\|\sum_{i=1}^{n} a_i \sigma_i\| := \sum_{i=1}^{n} \|a_i\|$ where σ_i are singular simplices & a_i are coefficients	$\|\nu\| := \int_X \nu^+ + \nu^-$ where $\nu := \nu^+ - \nu^-$ is the (unique) Hahn-decomposition of the signed measure ν into positive measures

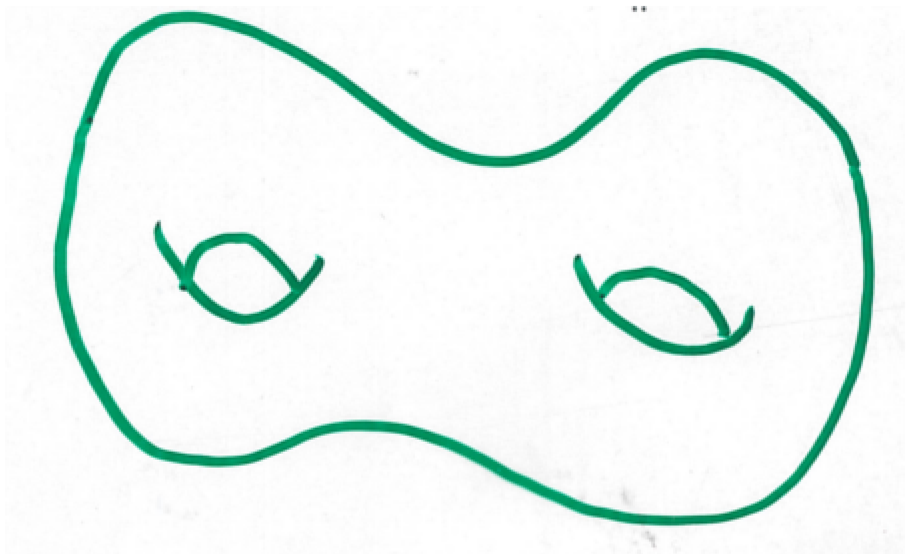
possibilities to represent the fundamantal cycle

The one-sphere:

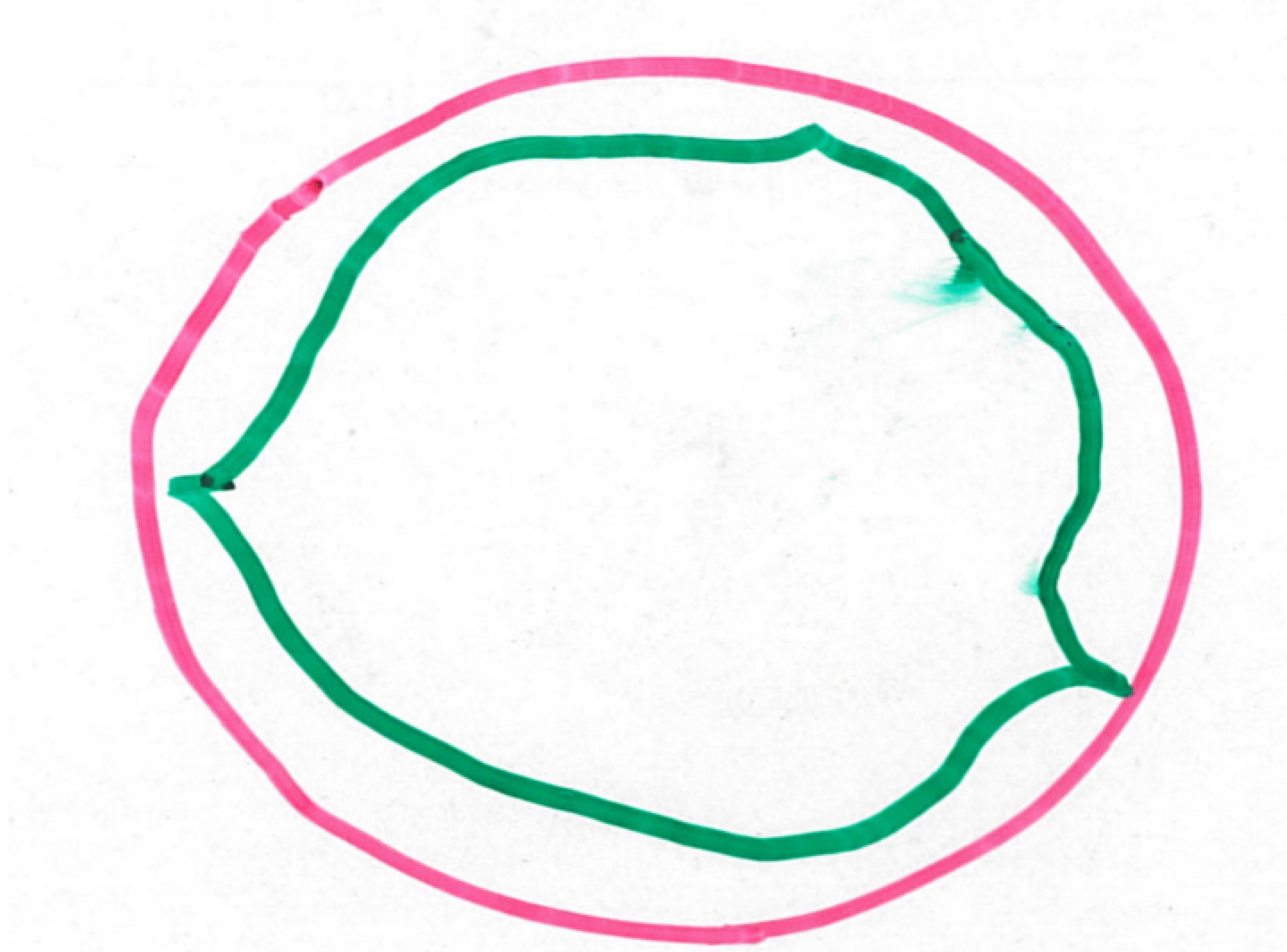




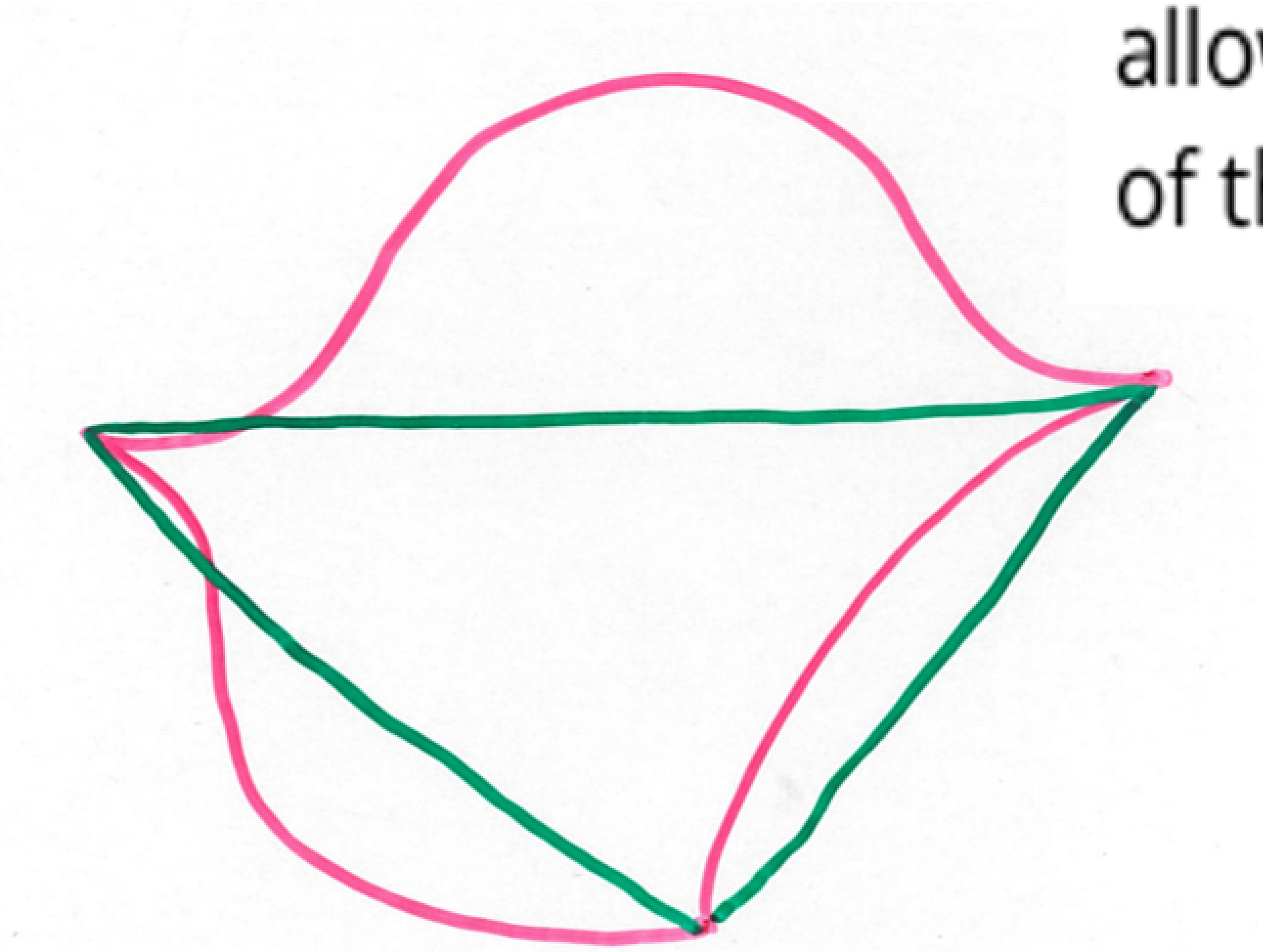
A przetzel :



Although a singular simplex can in principle cover arbitrary high hyperbolic volume



Thurston's process of straightening simplices



allows the statement, that w.l.o.g any simplex can cover at most the volume of the regular ideal simplex (:= v_n in dim. n).

Therefore $\|[M]\| = Vol(M)/v_n$

Literature on Milnor-Thurston homology theory:

1.) Thurston, W. P.: "Geometry and Topology of 3-manifolds", mimographed lecture notes, Princeton, 1977

2.) Gromov, Michael: "Volume and bounded cohomology", Inst. Hautes Études Sci. Publ. Math. No. 56 (1982), 5-99 (1983).

3.) Munkholm, Hans J.: "Simplices of maximal volume in hyperbolic space, Gromov's norm, and Gromov's proof of Mostow's rigidity theorem (following Thurston)", Topology Symposium, Siegen 1979 (Proc. Sympos., Univ. Siegen, Siegen, 1979), pp. 109-124, Lecture Notes in Math., 788, Springer, Berlin, 1980.

4.) Zastrow, Andreas: "On the (non)-coincidence of Milnor-Thurston homology theory with singular homology theory", Pacific J. Math., Vol. 186 (1998), no.\ 2, pp.\ 369--396.

5.) Hansen, Søren Kold: "Measure homology", Math. Scand. 83 (1998), no. 2, 205-219.

6.) Bowen, Lewis: "An Isometry Between Measure Homology and Singular Homology", arXiv:math/0401211, withdrawn due to an error.

7.) Soma, Teruhiko: "Volume of hyperbolic 3-manifolds with iterated pseudo-Anosov amalgamations", Geom. Dedicata 90 (2002), 183-200.

8.) Jungreis, Douglas: "Chains that realize the Gromov invariant of hyperbolic manifolds", Ergodic Theory Dynam. Systems 17 (1997), no. 3, 643-648.

9.) Kuessner, Thilo: "Efficient fundamental cycles of cusped hyperbolic manifolds", Pacific J. Math. 211 (2003), no. 2, 283-313.

10.) Ratcliffe, John G: One section in: "Foundations of hyperbolic manifolds" (Second edition), Graduate Texts in Mathematics, 149, Springer, New York, 2006.

11.) Löh, Clara: "Measure homology and singular homology are isometrically isomorphic", Math. Z. 253 (2006), no. 1, 197-218.

12.) Berlanga, Ricardo: "A topologised measure homology", Glasg. Math. J. 50 (2008), no. 3, 359-369.

13.) Frigerio, Roberto; Pagliantini, Cristina: "The simplicial volume of hyperbolic manifolds with geodesic boundary", Algebr. Geom. Topol. 10 (2010), no. 2, 979-1001.

14.) Frigerio, Roberto: "(Bounded) continuous cohomology and Gromov's proportionality principle", Manuscripta Math. 134 (2011), no. 3-4, 435-474.

15.) Frigerio, Roberto; Pagliantini, Cristina: "Relative measure homology and continuous bounded cohomology of topological pairs. Pacific J. Math. 257 (2012), no. 1, 91-130.

16.) Przewocki, Janusz: "Milnor-Thurston homology groups of the Warsaw Circle", Topology Appl. 160 (2013), no. 13, 1732-1741.

17.) Frigerio, Roberto: "A note on measure homology", Glasg. Math. J. 56 (2014), no. 1, 87-92

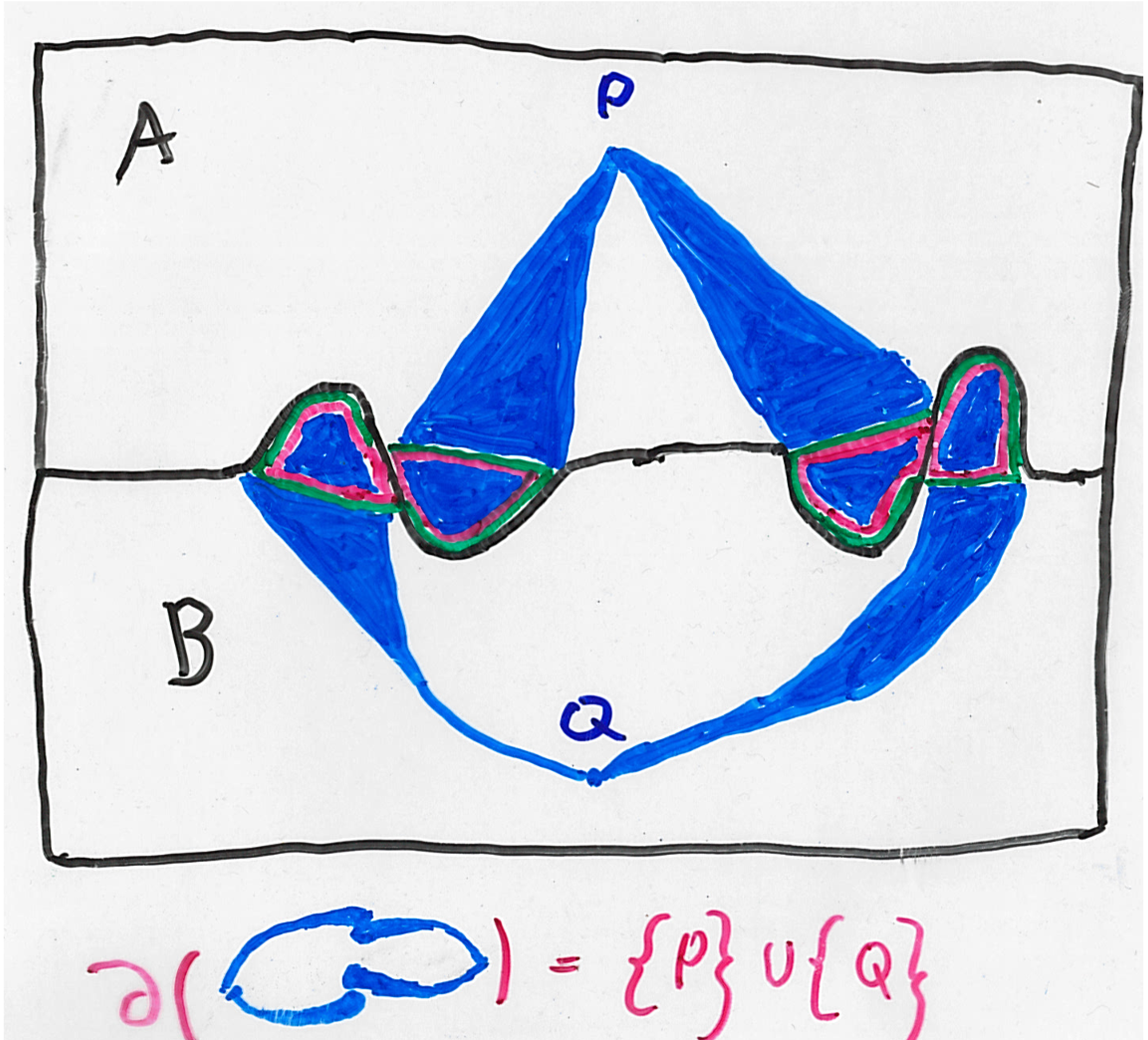
18.) Bader, U., Furman, A. & Sauer, R.: "Integrable measure equivalence and rigidity of hyperbolic lattices", Invent. Math. 194 (2013), no. 2, 313-379

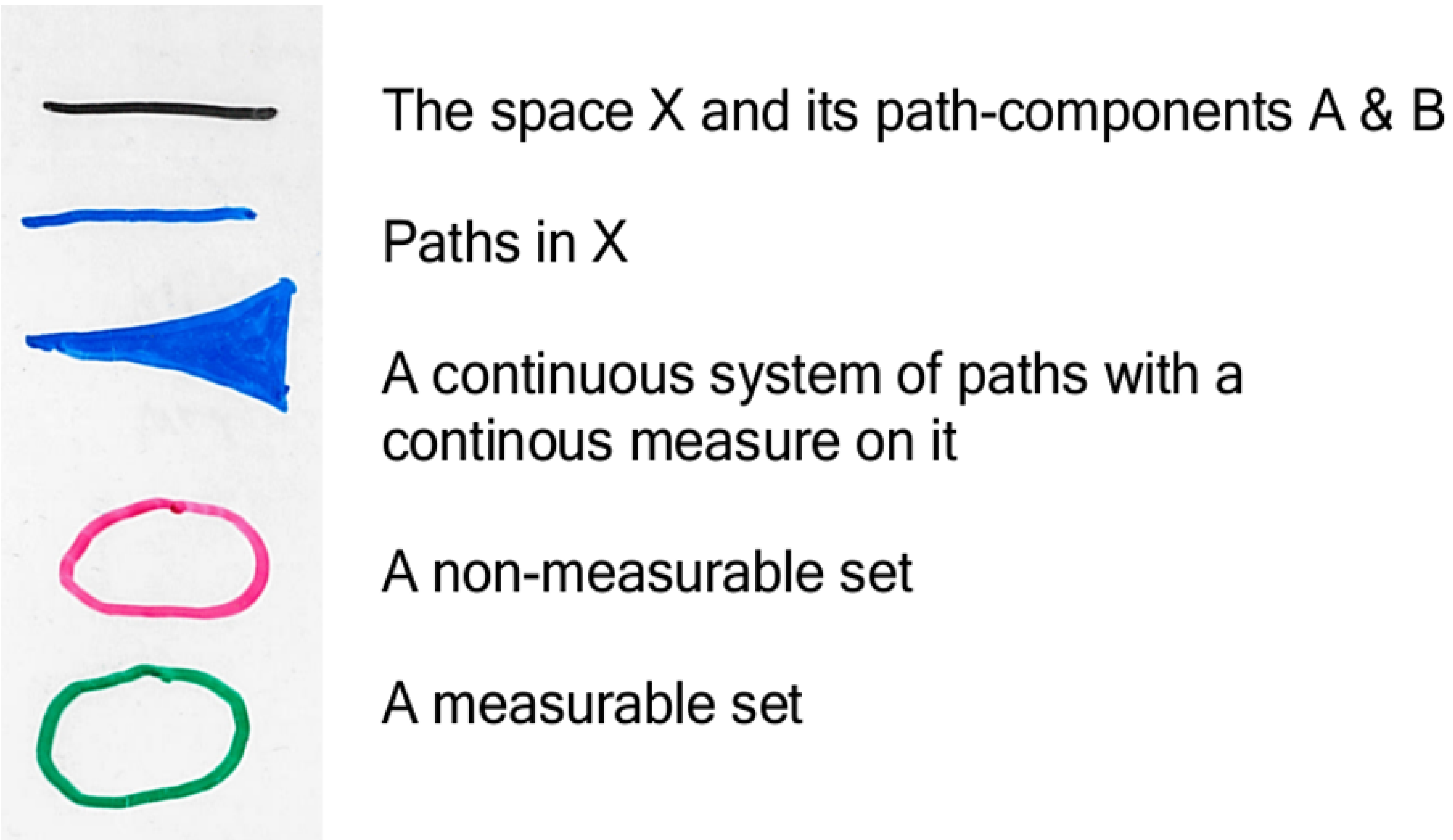
Since MTh-Groups are vector spaces, the only algebraic invariant is the dimension. Therefore it is often more interesting to ask:

- Is $H_k(X) \to \mathcal{H}_k(X)$ onto? - Or what is $\mathcal{H}_k(X) \setminus \mathrm{im}(H_k(X))$
- Is $H_k(X) \to \mathcal{H}_k(X)$ 1-1 ? - Or what are the cycles from the kernel?

This example is about to show, that injectivity does not automatically hold for this canonicak homomorphism, not even for the zeroth homology.

The space has two path-components, but with a measure cycle one can "walk" from one component to the other:

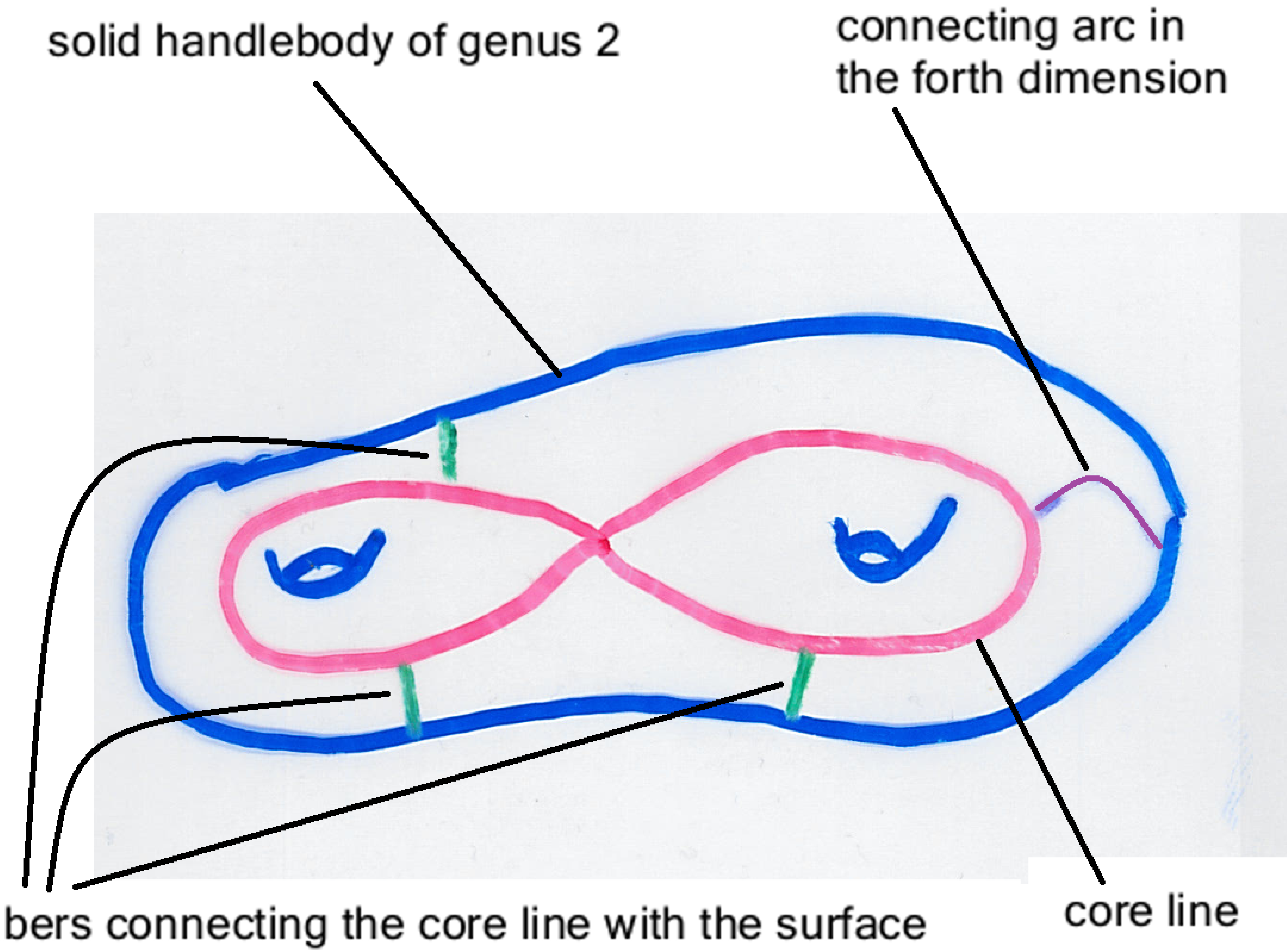




The preceding example was based on two aspects of non-triangulability

- non measurability
- a T_1-only topology. (comes in by additions using the cofinite topology to ensure copact carriers fr the measurs)

Please observe that our non-injectivity example easily extends to higher dimensions:



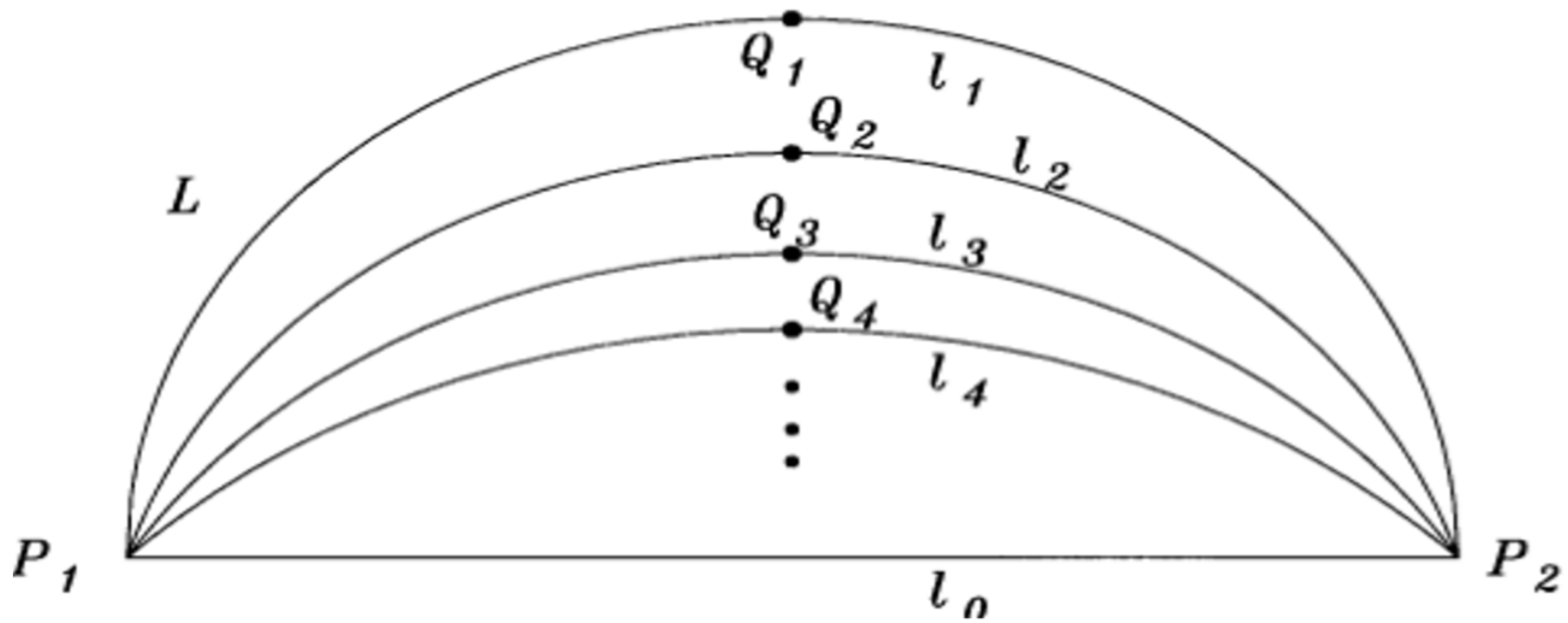
Replace in such a handlebody each fibre by our non-injectivity example.

We obtain: $H_2 = \mathbb{Z}$

and $\check{H}_2 = 0$

although:

- the fundamental cycle of H_2 has non-vanishing Gromov-Norm
- The space can be covered by finitely many contractible sets.

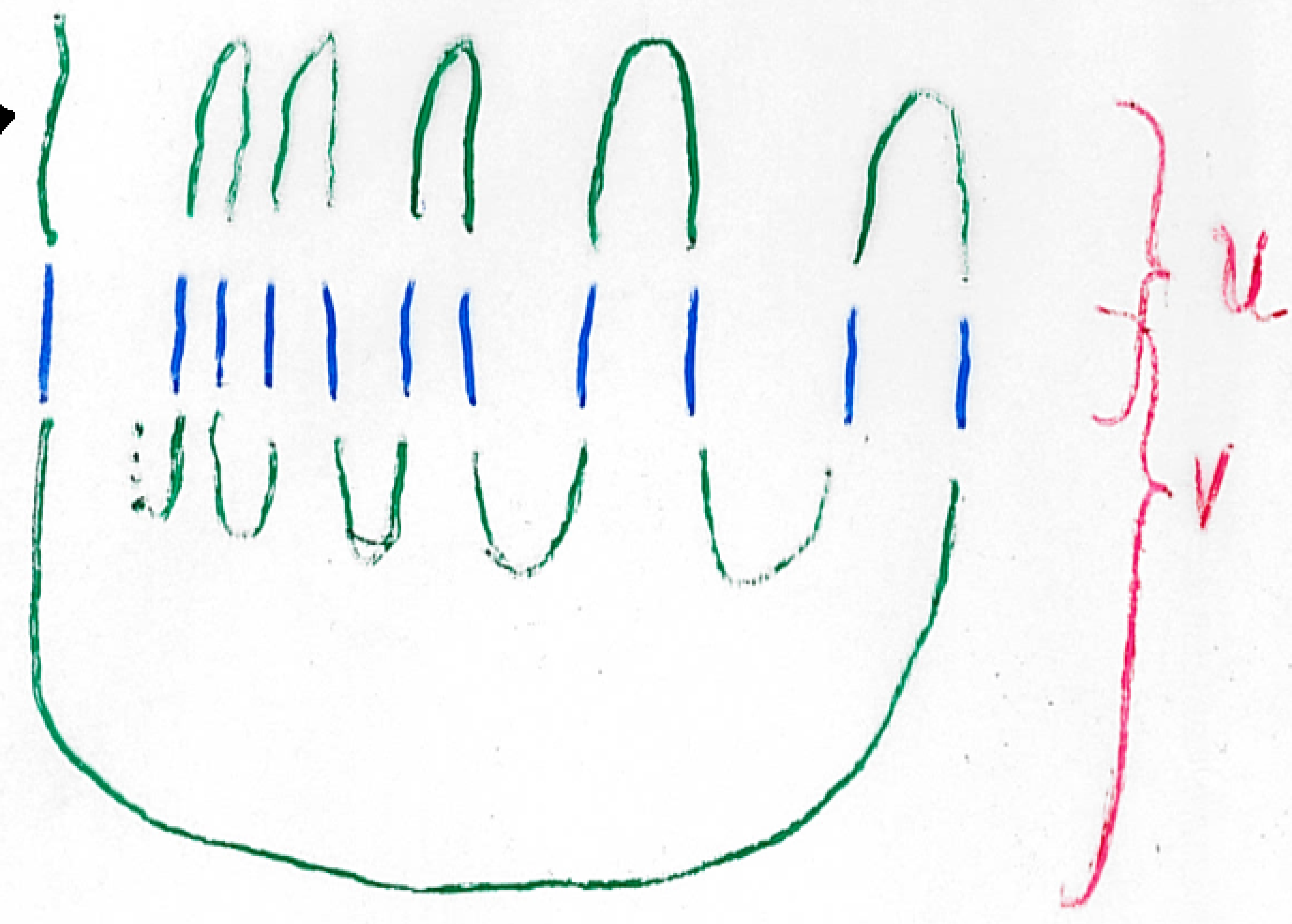


From [Z(MTh)] (1998) : The canonical homomorfism is not surjective.

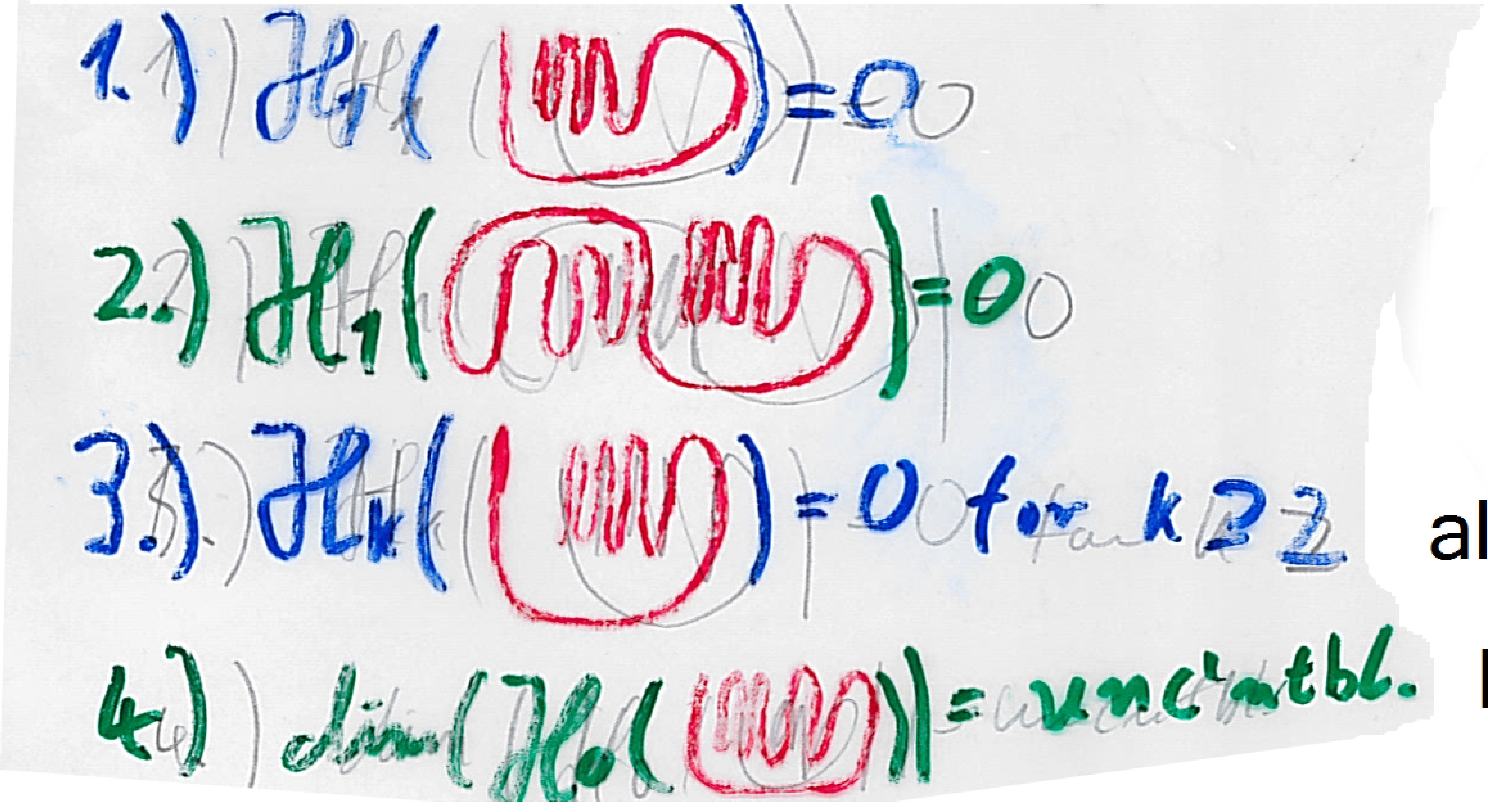
From [P(Dr)]: (2014) The groups are not even abstractly isomorphic:

(But in this case we have injectivity!)

Results of Przewocki for

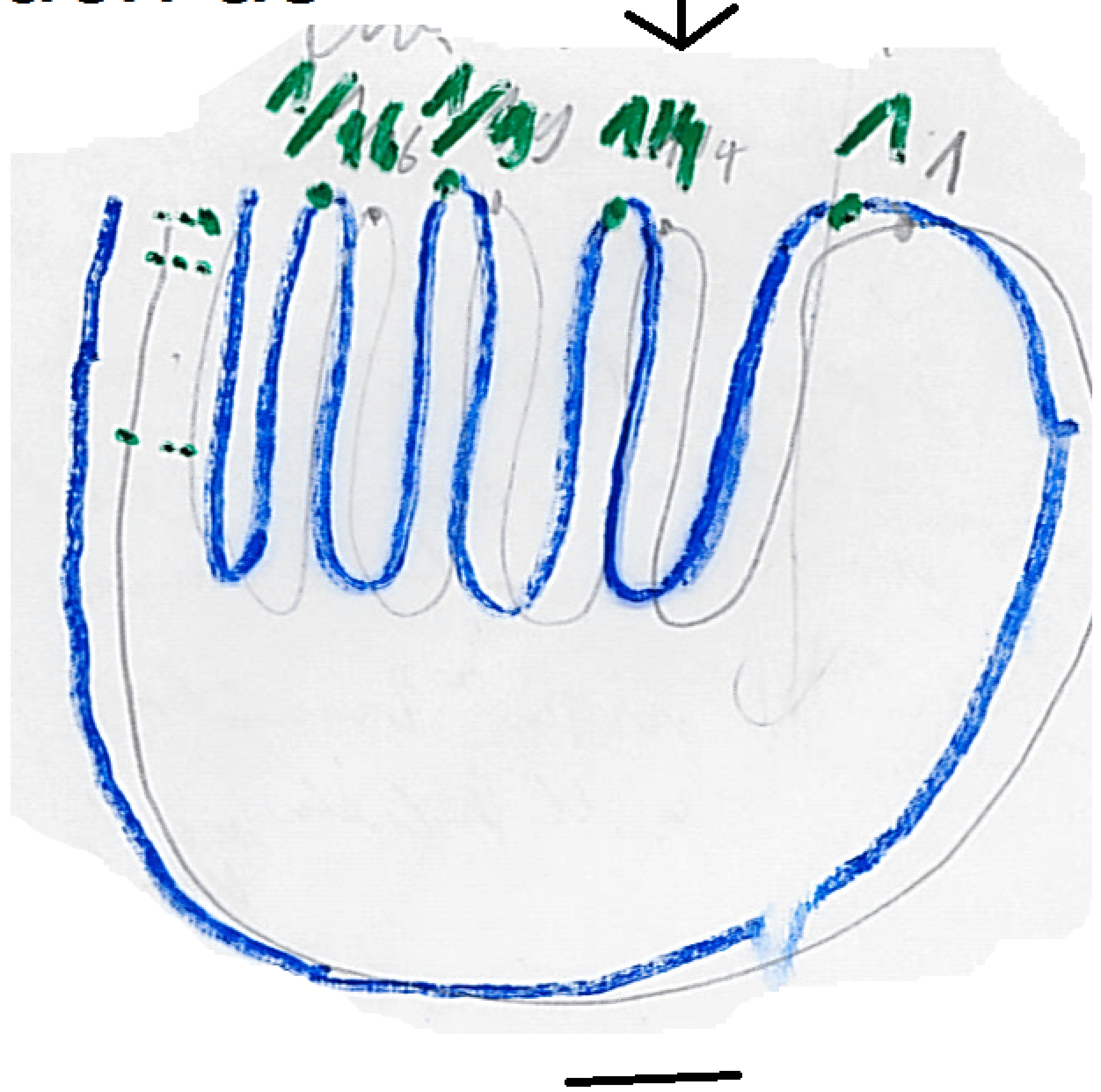


[P(Dr)], partially also [P(Publ)]



also for the spaces considered in 0.) & 2.)

because of cycles such as



(But in this case we have injectivity also!)

Further injectivity-results for $H_0(X) \rightarrow \mathcal{H}_0(X)$ by Przewocki

- For Peano-Continua (also "surjective") ([P(PhD)] and [PZ])
- for spaces with Borel-Path components (")
- for Polish spaces [not yet published]
  (involves the concept of universal measurablility, and that in this case the path components are universally measurable.

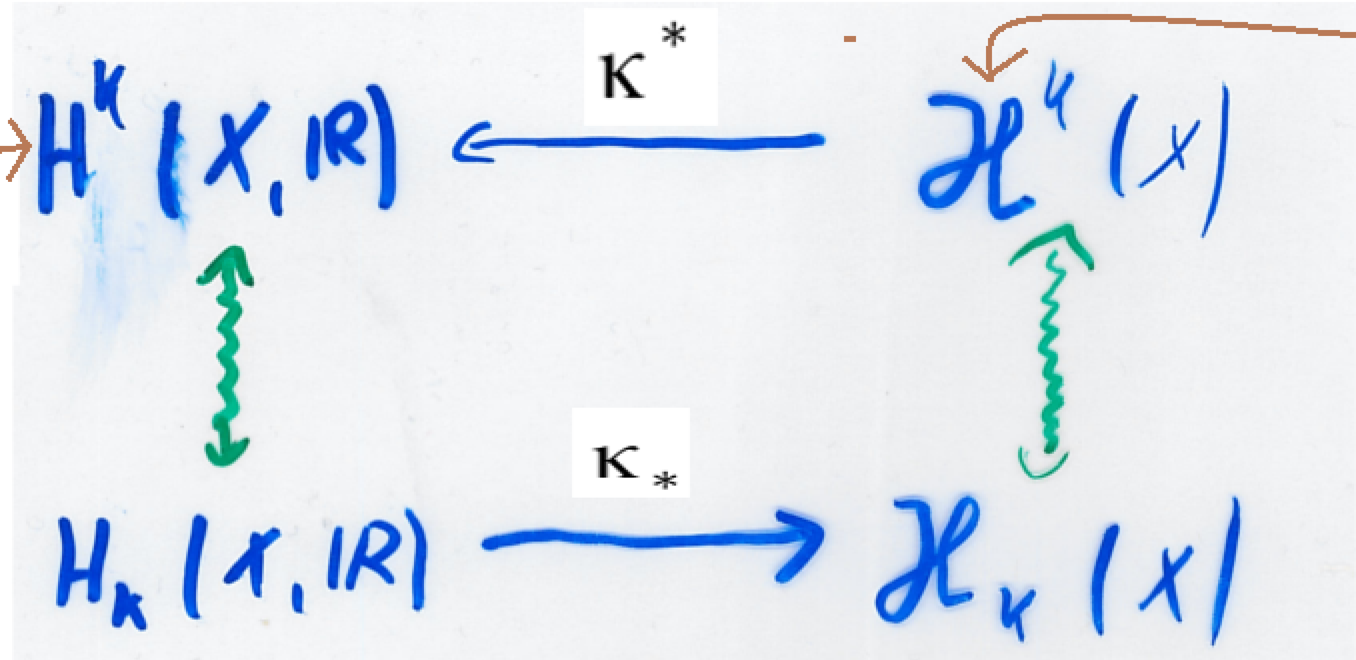
Although all currently known results for non-injectivity are based on non-measurable sets, the assumptions for our theorems are still quite restrictive:

Results from [KPZ]

(main initiator: Thilo)

The basic cocycle argument:

The canonical homomrphism κ_* & its dualization:



measurable cohomology

A standard diagramme-chase argument gives:

κ^* surj. $\Rightarrow$ κ_* inj.

Thm.0.1:

X 2^{nd} c'tbl T_1 -space, $\pi_1(X)$ c'tbl, X has a contr. generalized univ. covering space in the sense of [FZ]. $\Longrightarrow$ $\ker(\kappa_*)$ can contain only MTh-cycles with Gromov-Norm zero.

Via bounded cohomology and the fact that

$$C_b^k(\tilde{X}) = B(G, B(C((\Delta^k, v_0),(\tilde{X}, F)), \mathbb{R}))$$

Banach

strongly inj. module

Thm.0.2: X with c'ble π_k for all k

X has a gen. univ. cov. space in the sense of [FZ]

homotopy classes of simplices rel. vertices are Borel for X

X covered by finitely many Bprel sets whose closures are compact & contr. in X

$\Longrightarrow$ κ_* is injective.

The cocycle argument, together with a straightening, defined via acyclic models, for making an arbitrary cochain piecewise constant on Borel-sets

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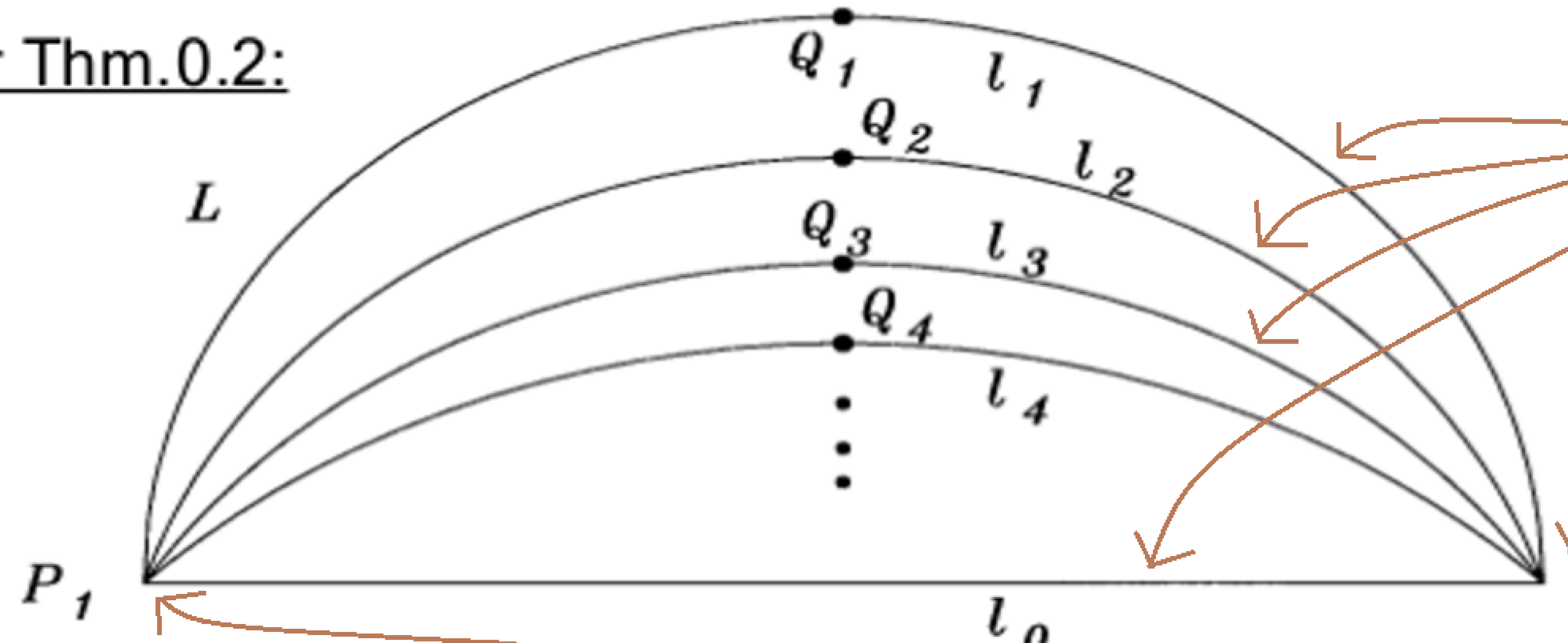
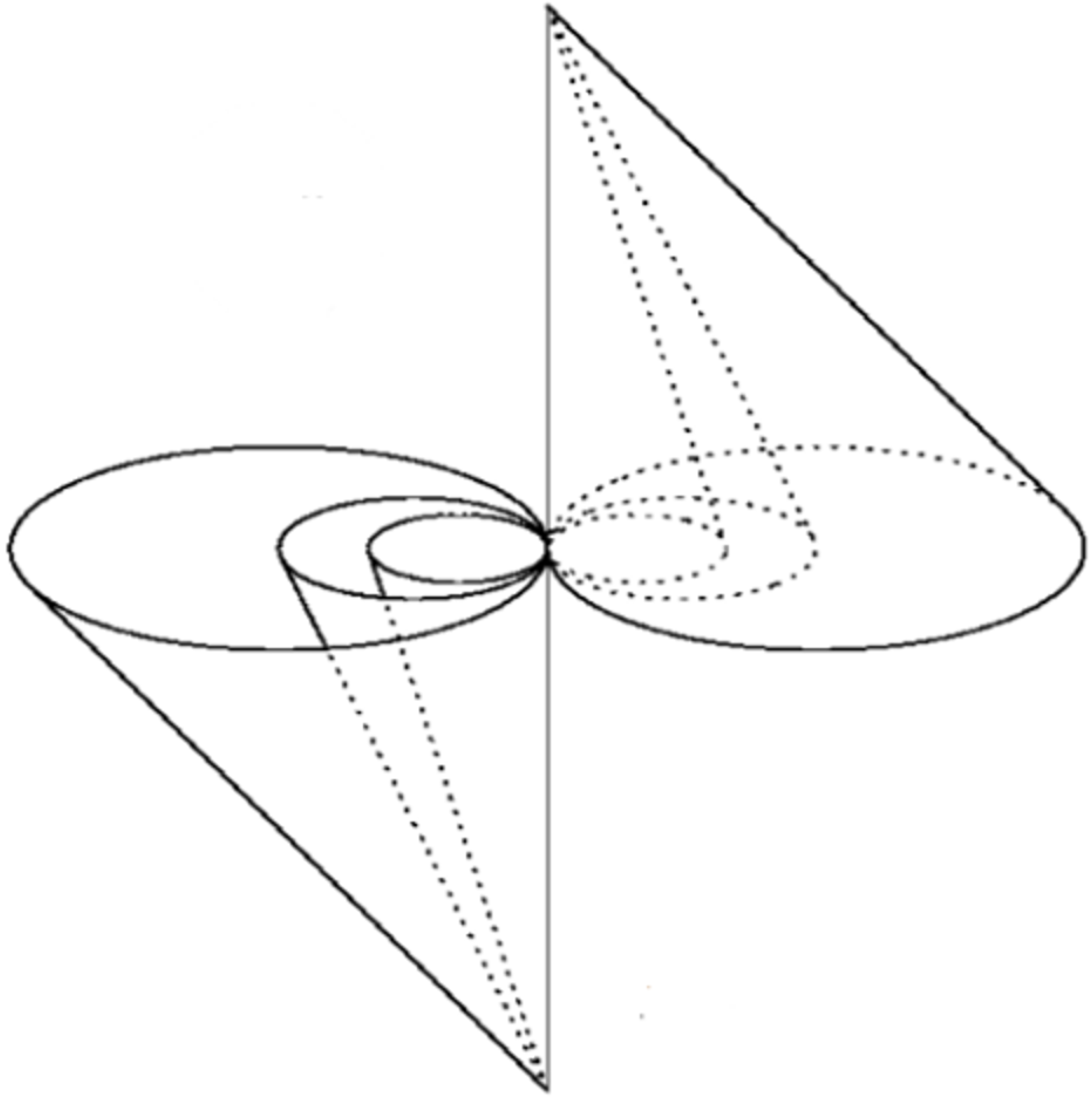
Example for Thm.0.2:



with Y = compact metric **triangulable space**

finitely many attaching points.

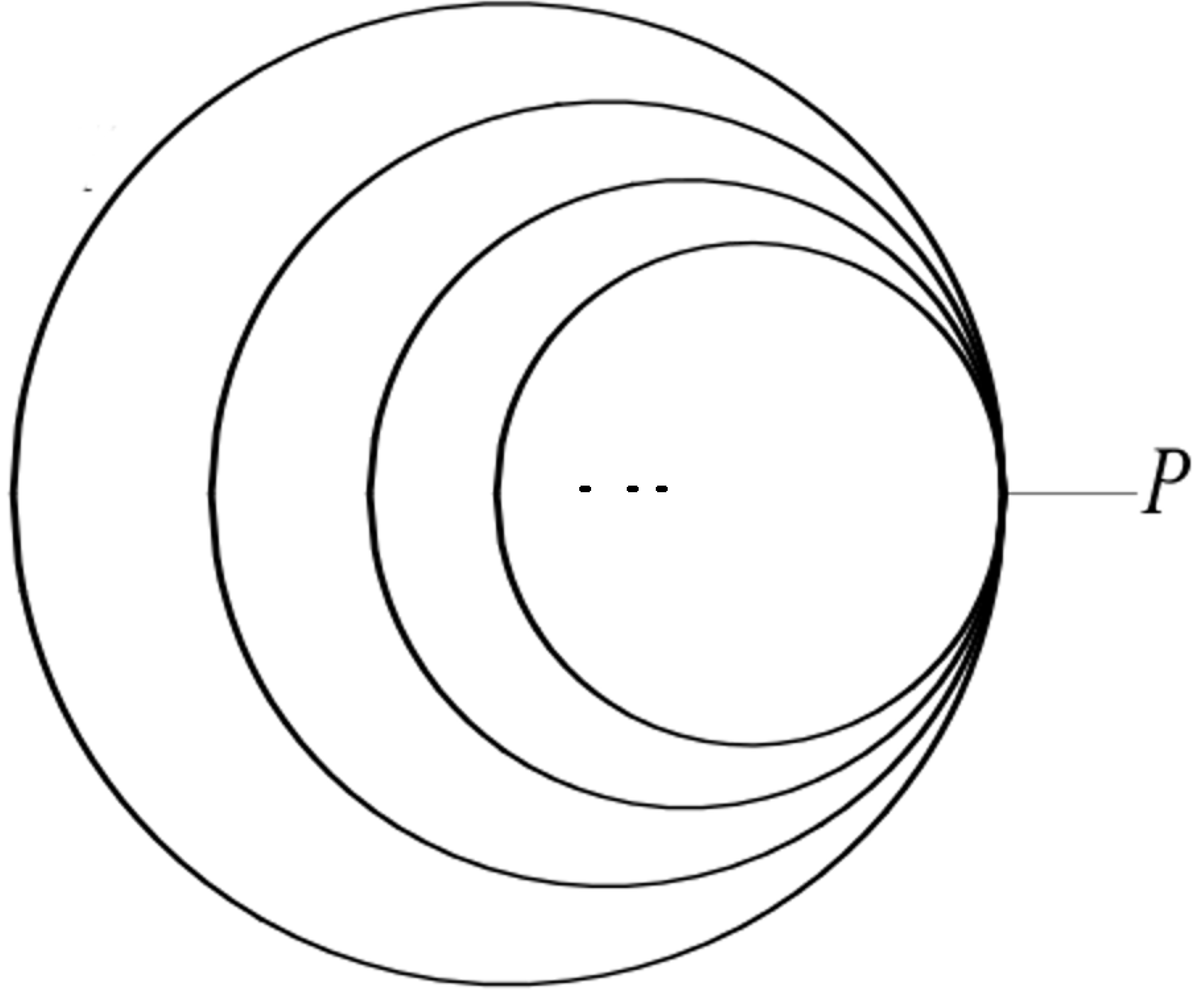


- All homotopy classes of paths in this space can be represented in an arbitrary small neighbourhood of the central point
- therefore the topology of π_1 is indiscrete, having only two open sets
- the same holds for the Borel-σ-algebra.
- But π_1 is uncountable
- Therefore the homotopy-classes will be non-(Borel)-measurable.

But this space is a Peano-continuum.

finite

CW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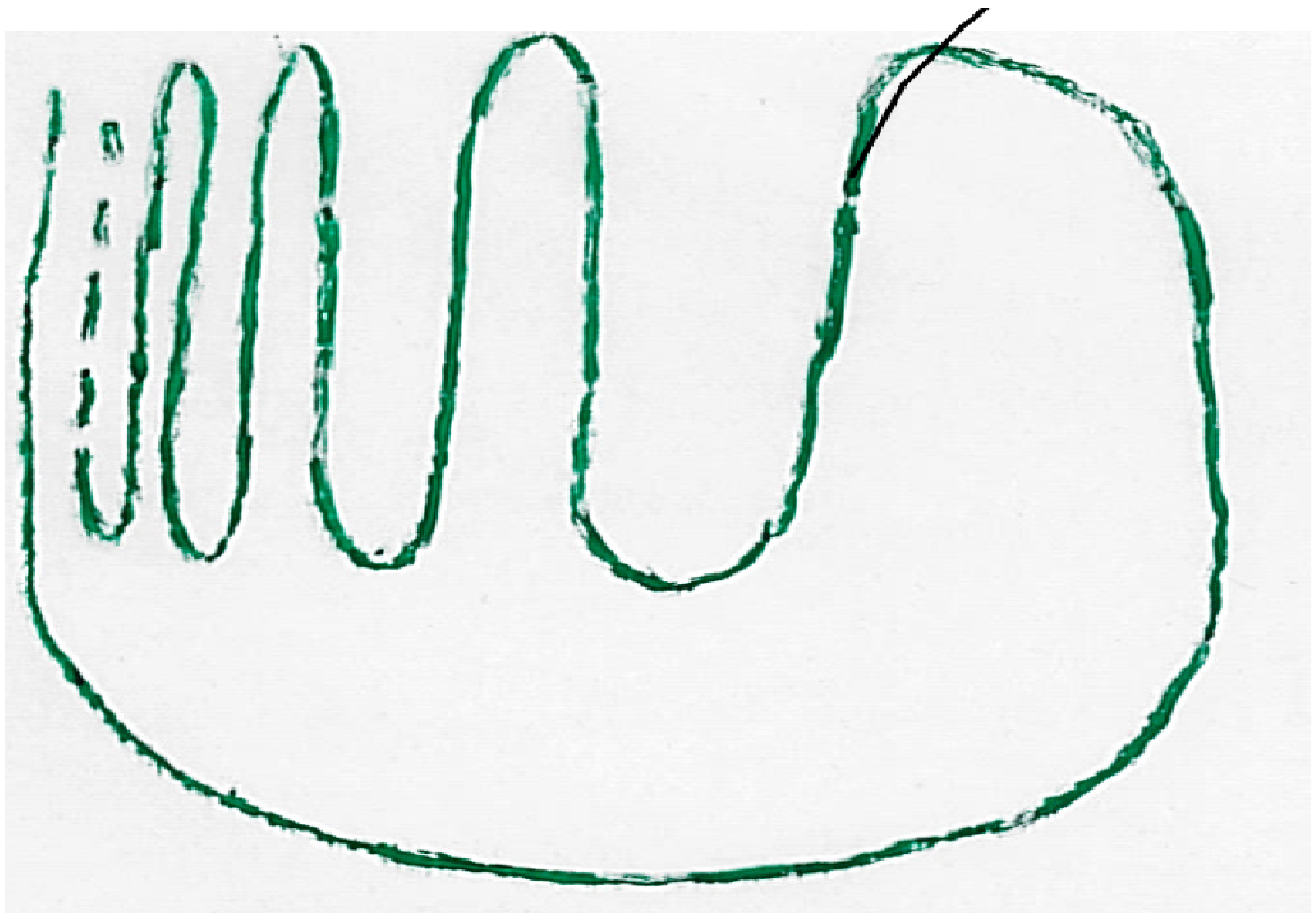
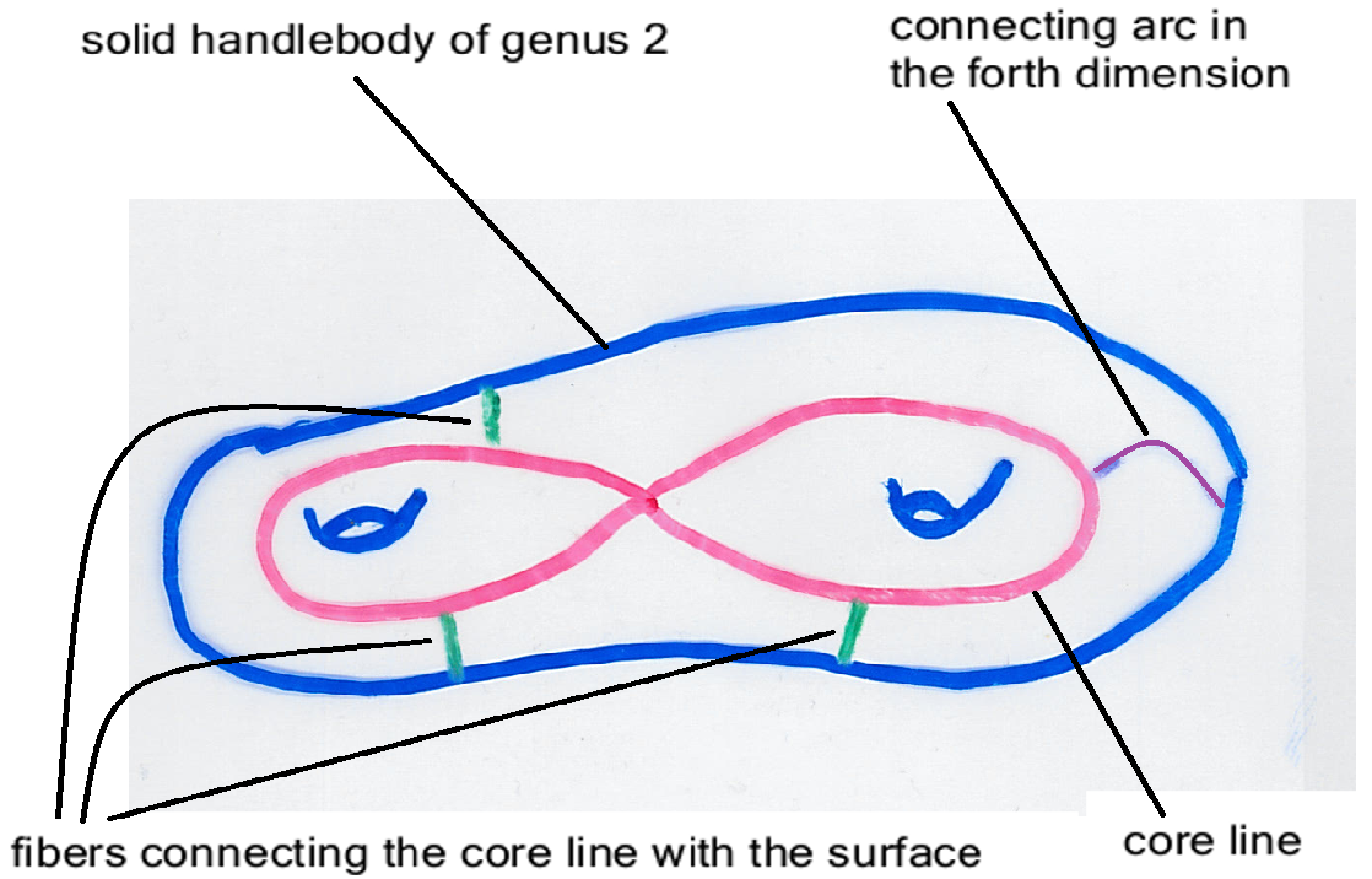



solid handlebody of genus 2
connecting arc in the forth dimension
fibers connecting the core line with the surface
core line

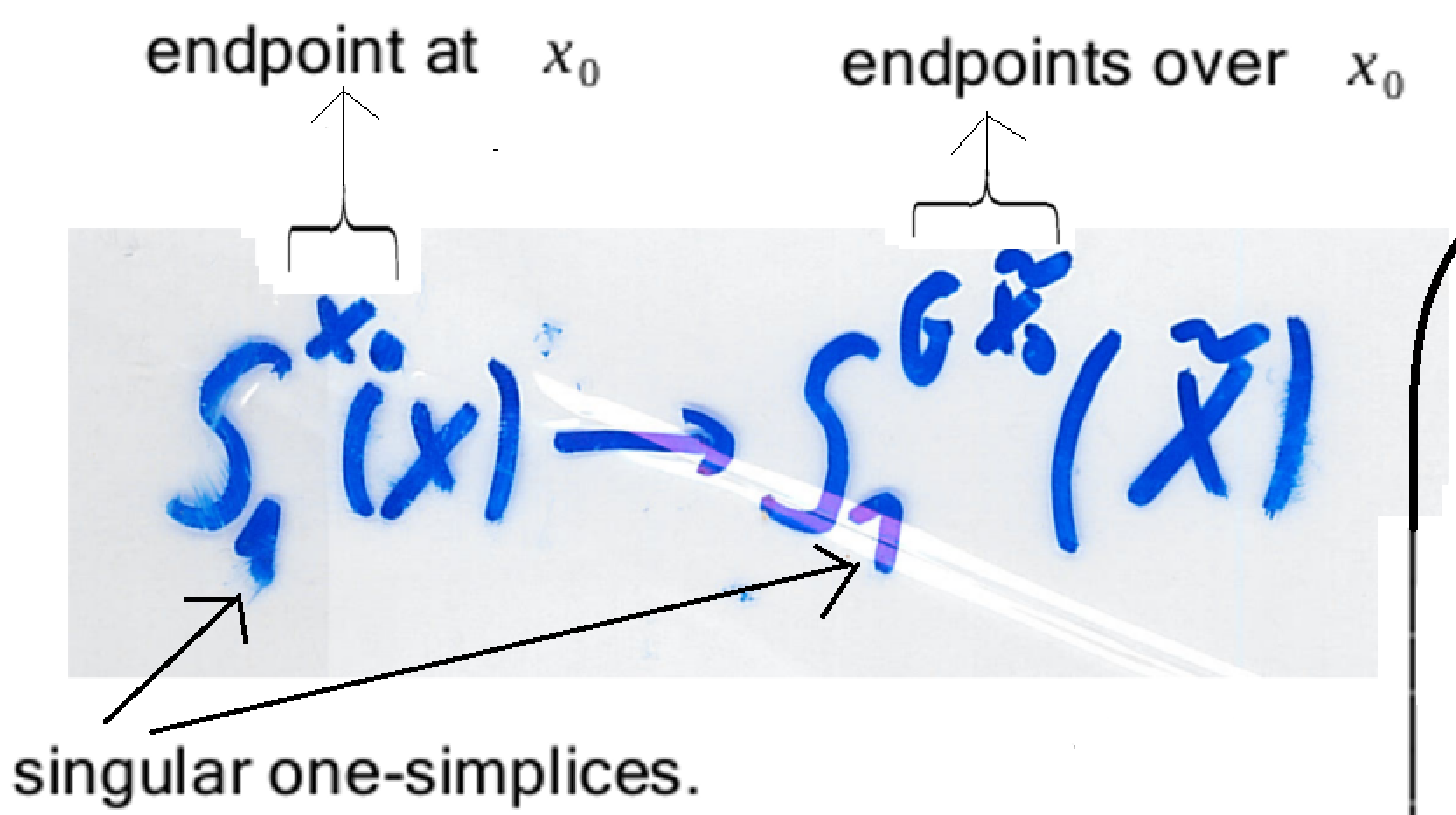
(To be published in [KPZ], Janusz' insight)

In Thm.0.2 the Assumption " π_1 c'tbl" was necessary, because for generalized covering spaces lifting of sets of paths may be non-Borel-measurable.

It must be so, if

- X is Polish
- semilocally simply connected at the base point x_0
- $\pi_1(X)$ unc'tbl.

The lifting of



cannot be Borel-measurable

Milnor-Thurston homology groups appear not to be the solution for wild spaces, because:

1.) They do not reflect an infinite shrinking-wedge structure of a space correctly (as seen for the Hawaiian Earrings)

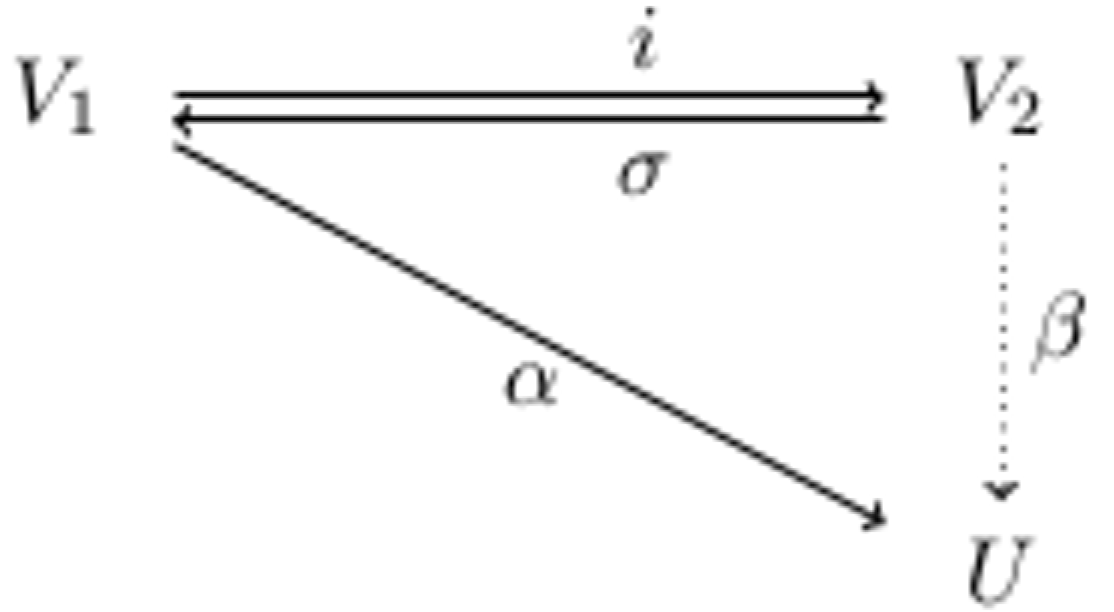
2.) They can behave very nasty, if spaces have connections, that that cannot be realized by paths

3.) ∃ uncountably many 0-1-sequences, and thus even for spaces with a countable structure ofter uncountably many alternatives occur. But measures have little compatibility with uncoutably unions (as seen for the Warsaw Circle).

⟹ Milnor-Thurston homology Theory will probably only be of little use in wild algebraic topology.

Thank you for your attention!

Definition 1.5. Let G be a topological group. A G-module U is called relatively injective if any diagram of the form



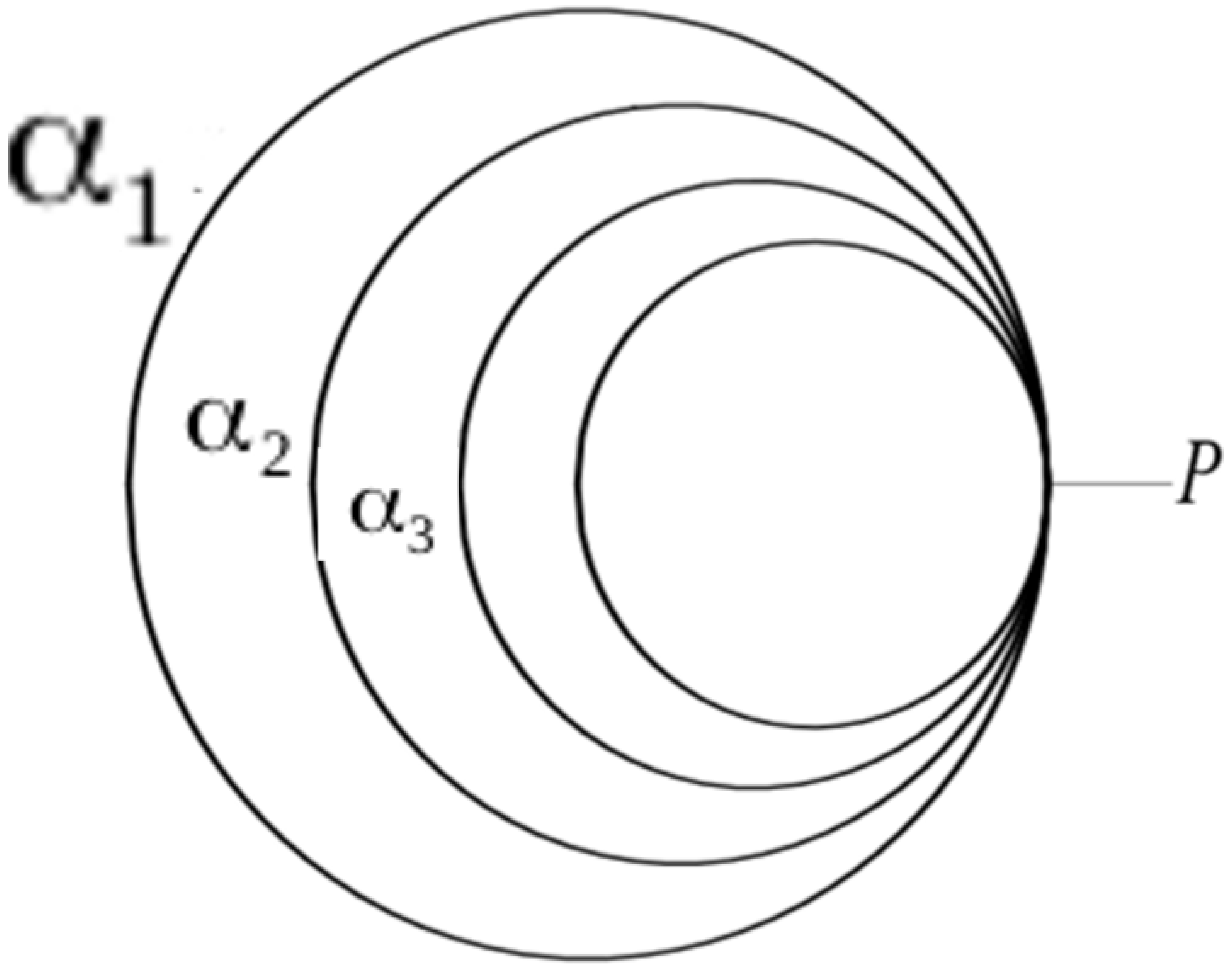
can be completed. Here $i: V_1 \to V_2$ is an injective morphism of G-modules, $\sigma: V_2 \to V_1$ is a bounded (not necessarily G-equivariant) linear operator with $\sigma \circ i = id$ and $\| \sigma \| \leq 1$, α is a G-morphism, and we want β to be a G-morphism with $\beta \circ i = \alpha$ and $\| \beta \| \leq \| \sigma \|$.

(3.2.2) LEMMA. For any Banach space V the G-module $B(G,V)$ is relatively injective. In particular, the G-modules $B(G^*)$ are relatively injective.

Proof. We consider the situation pictured in diagram (3.2.1), in which $\mathcal{U} = B(G,V)$ and we need to construct β, and all the rest is given. We define β by the formula $\beta(v)(g) = \alpha(\sigma(g \cdot v))(1)$. A calculation, which contrary to tradition we give, shows that β commutes with the action of G and that $\beta \circ i = \alpha$. Namely, $\beta(h \cdot v)(g) = \alpha(\sigma(g \cdot (h \cdot v)))(1) = \alpha(\sigma(gh \cdot v))(1) = \beta(v)(gh) = h \cdot \beta(v)(g)$ and $\beta(i(v))(g) = \alpha(\sigma(g \cdot i(v)))(1) = \alpha(\sigma(i(g \cdot v)))(1) = \alpha(\sigma \circ i(g \cdot v))(1) = \alpha(g \cdot v)(1) = g \cdot \alpha(v)(1) = \alpha(v)(g)$. Moreover, obviously $\|\beta(v)(g)\| = \|\alpha(\sigma(g \cdot v))(1)\| < \|\alpha(\sigma(g \cdot v))\| < \|\alpha\| \cdot \|\sigma\| \cdot |g \cdot v| \leqslant \|\alpha\| \cdot \|v\|$. It follows from this that β is a bounded operator and that $\|\beta\| \leqslant \|\alpha\|$, which finishes the proof.

The Hawaiian Earrings

$\mathbb{HE} = \mathbb{HE}^1 = \tilde{\bigvee}(S^1)$



- countably many circles
- whose radii are given by a null-sequence
- embedded as shown in the figure
- so that they have one common tangent point
- and topologized by this embedding.

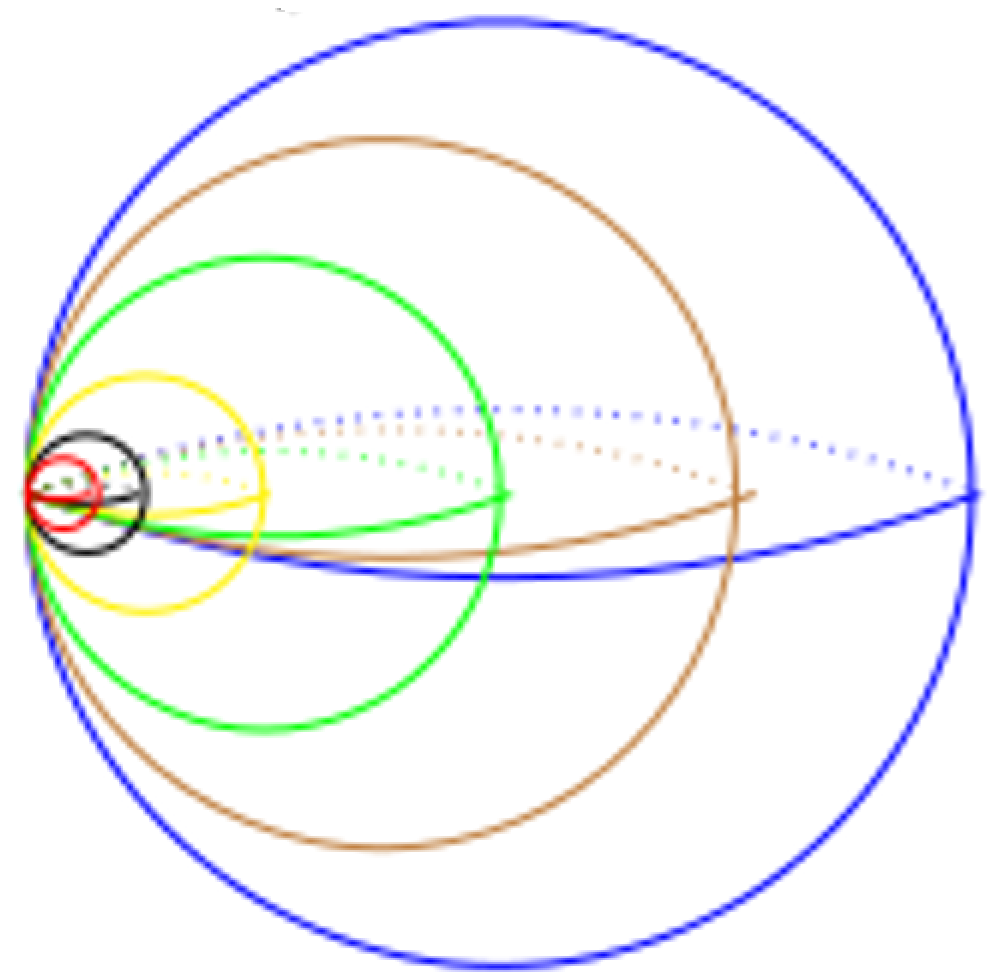
The analogous construction with 2-spheres embedded into R^3 is the so-called

Barrat-Milnor-Sphere

-or-

Two-dim. Hawaiian Earrings

$\mathbb{MB} = \mathbb{HE}^2 = \tilde{\bigvee}(S^2)$



The analogous abstract construction is called:

"A shrinking wedge"

(symbol: $\tilde{\bigvee}$)

The abstract definition of the topology of $\tilde{\bigvee}(X)$ demands, that a neighbouhood of the wedge-point has to contain almost all copies of the space X entirely.