Holonomy representation of Bieberbach group

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Crystallographic group

Let us denote by $E(n)$ the isometry group $Isom(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n$ of the n-dimensional Euclidean space.

Definition

A crystallographic group of dimension n is a cocompact and discrete subgroup of $E(n)$.

Example

1. Z^n

2. If $(B, \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}), (I, \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \in E(2)$, where $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then the group $\Gamma \subset E(2)$ generated by the above elements is a crystallographic group of dimension 2.

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Bieberbach theorems

The first part of the eighteenth Hilbert problem was about the description of discrete and cocompact groups of isometries of $\mathbb{R}^n$.

The answer for the above Hilbert problem was given by the German mathematician L. Bieberbach in 1913.

Theorem

(Bieberbach) *1. If* $\Gamma \subset E(n)$ *is a crystallographic group then the set of translations* $\Gamma \cap (I \times \mathbb{R}^n)$ *is a torsion free and finitely generated abelian group of rank n, and is a maximal abelian and normal subgroup of finite index.*

2. For any natural number n, there are only a finite number of isomorphism classes of crystallographic groups of dimension n.

3. Two crystallographic groups of dimension n are isomorphic if and only if they are conjugate in the group $A(n) = GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$.

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Flat manifold

Definition

A flat manifold M^n of dimension n is a compact connected Riemannian manifold without boundary with sectional curvature equal to zero.

Example

1. torus $\mathbb{R}^n/\mathbb{Z}^n \simeq \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n}$
2. $\mathbb{R}^n/\Gamma$, where $\Gamma \subset E(n)$ is a torsion free crystallographic group

Remark

Any flat manifold $M^n \simeq \mathbb{R}^n/\Gamma$, where $\Gamma = \pi_1(M^n)$. Γ is a torsion free crystallographic group of rank n.

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From the theorems of Bieberbach the fundamental group $\pi_1(M^n) = \Gamma$ (Bieberbach group) determines a short exact sequence

$$0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{p} G \to 0,$$

where $\mathbb{Z}^n$ is a torsion free abelian group of rank n and G is a finite group with is isomorphic to the holonomy group of M^n.

Collorary

Any flat manifold $M^n \simeq \mathbb{R}^n/\Gamma \simeq \mathbb{R}^n/\mathbb{Z}^n/\Gamma/\mathbb{Z}^n \simeq T^n/G$.

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Holonomy Representation

Let Γ be a crystallographic group. We have

$$0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{p} G \to 0.$$

Let $h_\Gamma : G \to GL(n, \mathbb{Z})$ be the integral holonomy representation defined by the formula

$$\forall_{g \in G} h_\Gamma(g)(e) = \bar{g} e \bar{g}^{-1},$$

where $\bar{g} \in \Gamma, p(\bar{g}) = g$ and $e \in \mathbb{Z}^n$. Since $\mathbb{Z}^n$ is a maximal abelian subgroup, h_Γ is a faithfull representation.

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E. Calabi

On the begining let us present some results about the relations between the properties of the holonomy representation h_Γ, the properties of Γ and the properties of the flat manifold $\mathbb{R}^n/\Gamma$. Most of them, in the language of the representation theory of the finite groups over real ($\mathbb{R}$) and rationals ($\mathbb{Q}$) numbers.

We start from some easy observation, which first was notices by E. Calabi in 1957

Proposition

Let Γ be any Bieberbach group then the following conditions are equivalent:

(i) the centre $Z(\Gamma)$ is trivial,

(ii) $(\Lambda)^G$ is trivial,

(iii) the first Betti number of a flat manifold $\mathbb{R}^n/\Gamma$ is zero.

Proof: The equivalence (i) and (ii) is obvious and follows from definition. The first Betti number of a manifold $\mathbb{R}^n/\Gamma$ is equal to $\dim_{\mathbb{Q}} \Gamma/[\Gamma, \Gamma] \times \mathbb{Q}$. We have to mention that the above observation can be consider as a method of the classification of flat manifolds by induction.

Anosov

In 1972 (Topology) Porteus proved very nice result about the existance of Anosov diffeomorphisms on flat manifolds $M = \mathbb{R}^n/\Gamma$. The diffeomorphism $f : M \to M$ is Anosov if any eigenvalue λ of differentaial $df : T_xM \to T_xM$ has the norm $\mid \lambda \mid \neq 1$.

Theorem

Let M be a flat manifold with fundamental group Γ. Then the following conditions are equivalent:

(i) M support an Anosov diffeomorphism,

(ii) any $\mathbb{Q}$-irreducible component of the holonomy representation h_Γ of multiplicity one is $\mathbb{R}$-reducible.

Conjecture

The above result for infranilmanifolds (almost flat manifolds) still is an open problem. That is related to the well know conjecture of not existing of the Anosov diffeomorphisms on manifolds different from infranilmanifolds.

The theorem of Porteus was for us the main inspiration to consider affine symmetries of flat manifolds.

Symmetries

Theorem

Let Γ be a Bieberbach group. Then the following conditions are equivalent.

(i) Outer automorphism group $Out(\Gamma)$ is finite,

(ii) the holonomy representation ϕ_Γ

is $\mathbb{Q}$-multiplicity free and each $\mathbb{Q}$-irreducible component is $\mathbb{R}$-irreducible.

Proof It is not difficult to see that Out(Γ) is finite if and only if centralizer of the subgroup $\phi_\Gamma(G)$ in $GL(n, \mathbb{Z})$ is finite. We would like to mention that the result about finitness of the centralizer of the finite in $GL(n, \mathbb{Z})$ was already consider by Siegel. Next the crucial is following lemma and analisis of the relation of irreducible representations over $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$.

Lemma Let S be a rational representation of a finite group of dimension m. Then S commutes with an element $K \in GL(n, \mathbb{Z})$ of infinite order if and only if S commutes with an element $K_1 \in GL(m, \mathbb{Q})$ of infinite order whose characteristic polynomial has integer coefficients with a unit constant term.

That result was generalized.

Theorem

Let Γ be a Bieberbach group.

Then the following conditions are equivalent.

(i) Outer automorphism group $Out(\Gamma)$ is polycyclic by finite,

(ii) the holonomy representation h_Γ is $\mathbb{Q}$-multiplicity free and each $\mathbb{R}$-reducible component has the Schur index$_{\mathbb{Q}}$ one.

We only mention that the crucial point in the proof is an application of the result of H. Zassenhaus about the unit group of a Dedekind order. In 1991 F.Johnson and E. Rees proved the following.

Kähler/Symplectic

Theorem

Let Γ be a Bieberbach group. Then the following conditions are equivalent.

(i) $\mathbb{R}^n/\Gamma$ is a Kähler (complex) flat manifold

(ii) n is an even number and each $\mathbb{R}$ and $\mathbb{C}$-irreducible component of the holonomy representation h_Γ occurs with even multiplicity.

Table

Let us summarize all the above results in the table.

Year	Group Γ	Holonomy representation ϕ_Γ
1956	$Z(\Gamma) \neq 0$	$(\Lambda)^G$ is not trivial
1972	$\mathbb{R}^n/\Gamma$ s.Anos.diff.	Each $\mathbb{Q}$-irr. sum. of mult. one is $\mathbb{R}$-reducible
1991	$\mathbb{R}^n/\Gamma$ is Kähler	n is even and each $\mathbb{R}$-irr. and $\mathbb{C}$-irr. sum. occurs with even multipl.
1996	$\mathrm{Out}(\Gamma)$ is finite	$\mathbb{Q}$-multipl. free and each $\mathbb{Q}$-irr. component is $\mathbb{R}$-irr.

table

Year	Group Γ	Holonomy representation ϕ_Γ
2003	$\text{Out}(\Gamma)$ is p.by f.	$\mathbb{Q}$-multipl. free and the Schur *index*$_{\mathbb{Q}}$ of each $\mathbb{R}$-reducible sum. is equal to one.

Hiss-Szczepański

Moreover we have also another results about relations between the holonomy representation and properties of the crystallographic (Bieberbach) groups. So far there are not such spectacular as above Let us start from the following deep theorem.

Theorem

Let $\phi : G \to GL(n, \mathbb{Z})$ be an integer representation of finite group G which is $\mathbb{Q}$-irreducible, then in the second cohomology group $H^2(G, \mathbb{Z}^n)$ does not exist an element which correspond to torsion free crystallographic group.

Conjecture II

Conjecture (S. 2003) For any finite group G there exists a Bieberbach group Γ with $\mathbb{Q}$-multiplicity free holonomy representation.

Comments 1. For crystallographic groups the above conjecture is trivial.

2. We can prove it for some abelian groups, p-groups, dihedral groups and simple groups. Moreover, for any finite group G there exist a Bieberbach group with the first Betti number one.

3. Assume that the above conjecture is true, then if the Whitehead group $\mathrm{Wh}(G)$ is finite then, G is the holonomy group of a Bieberbach group with finite outer automorphism group.

$\mathbb{R}$-irreducible representations

Consider the decomposition of the holonomy representation h_G into $\mathbb{R}$-irreducible components $V_1, V_2, \ldots, V_k$. There are three kinds of summand: absolutely irreducible $End_G(V) \simeq \mathbb{R}$, complex $End_G(V) \simeq \mathbb{C}$ and quaternionic $End_G(V) \simeq \mathbb{H}$. A irreducible

Examples

- Let Γ be a Hantzsche-Wendt Bieberbach group of dimension n. All $\mathbb{R}$-irreducible component of the holonomy representation h_Γ of Γ are absolutely irreducible (have real type).
- The eight dimensional Bieberbach group with holonomy group $\mathbb{Z}_3 \times \mathbb{Z}_3$ with the trivial center all $\mathbb{R}$ irreducible components of the holonomy representation are complex.

We define a Bieberbach group with *complex type* holonomy representation. Let a, b denote generators of $\mathbb{Z}_3 \times \mathbb{Z}_3$. The holonomy representation is as follows:

$$a \mapsto A, b \mapsto B,$$

where $A = \text{diag}(1, X, X, X)$ and $B = \text{diag}(X, 1, X, X^2)$. Moreover $X = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$.

Question/Remark

Up till know we do not know example of the Bieberbach group with the trivial center and holonomy representation of quaternionic type.