BOŻENA PIĄTEK

AUTOREFERAT
(wersja w języku angielskim)

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Contents

1 Personal date

Names, surname	Bożena Piątek
Date and place of birth	25.07.1979, Knurów

2 Degrees

- **M. Sc., November 2002**
  Faculty of Mathematical and Physical Sciences, Silesian University of Technology,
  thesis entitled: An application of thermal polynomials in inverse heat conduction problem
  supervisor: prof. dr hab. inż. Radosław Grzymkowski

- **Ph.D. in Mathematics, April 2007**
  Faculty of Mathematical Physical and Technical Sciences, Pedagogical Academy in Kraków,
  thesis entitled: An application of Riemann integral of multifunctions in functional equations and inclusions and its relation to Aumann integral
  supervisor: dr hab. Wilhelmina Smajdor

3 Employment

- October 2007 – June 2008 assistant in Institute of Mathematics, Silesian University of Technology

- July 2008 – adjunct in Institute of Mathematics, Silesian University of Technology

- February 2011 – April 2011 postdoctoral fellow, Instituto de Matemáticas Universidad de Sevilla

4 Scientific achievement

(w rozumieniu art. 16 ust. 2 Ustawy z dnia 14 marca 2003 r. o stopniach naukowych i tytule naukowym oraz o stopniach i tytule w zakresie sztuki)

My scientific achievement is the monothematic publication cycle entitled

“ The fixed point property in unbounded and nonconvex subsets of geodesic spaces”

consisting of the following articles

[P01] R. Espínola and B. Piątek, *The fixed point property and unbounded sets in CAT(0) spaces*, J. Math. Anal. Appl. 408 (2013), 638–654.

[P02] B. Piątek and R. Espínola, *Fixed points and non-convex sets in CAT(0) spaces* Topol. Methods Nonlinear Anal. 41(1) (2013), 135–162.

[P03] B. Piątek, *The fixed point property and unbounded sets in spaces of negative curvature*, Israel J. Math. 209 (2015), 323–334.

[P04] B. Piątek, *On the fixed point property for nonexpansive mappings in hyperbolic geodesic spaces*, J. Nonlinear Convex Anal. 19(4) (2018), 571–582.

[P05] B. Piątek, *The behavior of fixed point free nonexpansive mappings in geodesic spaces*, J. Math. Anal. Appl. 445 (2017), 1071–1083.

4.1 Introduction

Although CAT(κ) spaces were introduced in the eighties, at the beginning, because of their relation with hyperbolic groups, the main attention was paid to the study of their isometries. At the turn of the century W. A. Kirk gave a cycle of seminar talks during which he presented a number of basic results devoted to several types of problems related to fixed points of nonexpansive mappings defined on these spaces. The variety of such results and the open questions raised by the author (see [31] and [32]) have motivated the study in the settings of CAT(0) spaces of mappings that are not necessarily isometries. The problems and open questions discussed during the above mentioned talks can be grouped in three groups.

The first one concerns the existence and a localization of fixed points for mappings defined on bounded sets. Here the fundamental result is the following one:

Theorem 4.1. *([31] Theorem 18)*
Let X be a complete and bounded CAT(0) space. Then each nonexpansive mapping $T\colon X \to X$ has at least one fixed point.

This result was later generalized by the same author to the case of non-convex bounded subsets (see [30, Theorem 3.3]) and by R. Espínola and A. Fernández-León to the case of CAT(κ) spaces with positive κ (see [17, Theorem 3.9]). Afterwards the same problem was studied in other more general classes of spaces (see for instance [P13] and [44]).

The second group of problems is devoted to the relation between the existence of fixed points and geometric properties of the space. In this way one may consider mappings defined on non–convex or unbounded spaces (in the case of nonexpansive mappings) as well as connections with compactness for continuous mappings. My main research during the last years has focused magnificently on these problems.

The most important contribution of my authorship is the full characterization of boundedness of CAT(0) spaces via the fixed point property of nonexpansive mappings (see Theorems 4.28 and 4.32). At the same time it is worth mentioning that I also obtained a wide range of properties of fixed point free nonexpansive mappings defined on unbounded spaces. Let me also remark that in the paper [P19] written in cooperation with Prof. Genaro López-Acedo we gave a similar characterization via the fixed point property for continuous mappings. The research methods that were applied in these two cases are completely different. In this report the fixed point property of a space X means that for each nonempty closed and convex (not necessary bounded) subset $C \subset X$ and $T\colon C \to C$ the mapping T has at least one fixed point.

Finally, let us mention that the third group of problems deals with the approximation of fixed points in CAT(κ) spaces. While the approximation for CAT(0) spaces does not require the development of new techniques the situation is different in the more general case of CAT(κ) spaces and the first papers devoted to this problem are of my authorship.

The report is organized as follows. First we collect basic definitions and relations of geodesic spaces needed in the further exposition. Next we recall some known results about the existence of fixed points for unbounded and non-convex CAT(0) spaces. Notice that so far the behavior of mappings defined on non-convex domains has been understood much better than in the case of unbounded domains. We come back to this problem at the beginning of Section 4.3 where we give the full picture of this situation. The rest of this section contains my contribution to fixed point theory in the unbounded, as well as the non-convex subsets of geodesic spaces.

In Section 5 we collect results that are not included in the main scientific achievement, although most of them are also related to fixed point theory in geodesic spaces. First we introduce the methods used to approximate fixed points in CAT(κ) spaces and their influence on the further research. Next we mention Klee's result on the characterization of sets with the fixed point property for continuous mappings defined on closed convex subsets of linear spaces using their geometric structure. More precisely, we are interested in the compactness of the set and the existence of closed topological rays. In Section 5.2 we show how far the counterpart of Klee's result works in the case of geodesic spaces. The third collection of papers described in more detail is devoted to the concept of diversity and hyperconvex spaces. I decided not to discussed all my papers and chose only the most relevant directions of my research.

4.2 Preliminaries

We begin with some standard notation and basic facts about geodesic spaces, which we use in the sequel; one may find a much more thorough description of the concepts presented below, for example, in [11], [13] and [50]. Let us assume that a metric space (X, ρ) is geodesic which means that each couple of points can be joined by a geodesic, i.e., an isometric embedding $\gamma\colon [0, d(x,y)] \to X$ such that $\gamma(0) = x$, $\gamma(d(x,y)) = y$. The image of γ is called a geodesic segment and if it is unique is denoted by $[x, y]$. If the geodesic can be isometrically extended to $[0, \infty)$, we say that the image $\gamma([0,\infty))$ is a geodesic ray. We say that a subset A of a geodesic space X is convex if for all $x, y \in A$ each geodesic segment joining these two points also belongs to A. Moreover, a metric space where every two points are joined by a unique geodesic is called uniquely geodesic.

Now we focus on the concept of $\mathrm{CAT}(\kappa)$ spaces. This type of spaces was introduced in the eighties by M. Gromov and has been extensively studied in last years by a number of authors (see, for example, [5], [8], [11], Chapter 9 in [13] and also Chapter 9 in [33]). On the one hand, $\mathrm{CAT}(\kappa)$ spaces share a lot of useful properties with Riemannian manifolds as well as Hilbert spaces (under the additional assumption $\kappa \leq 0$). On the other hand, they generalize a large class of more specific spaces, which have been studied so far separately. Typical examples of $\mathrm{CAT}(\kappa)$ spaces include, among others, above mentioned manifolds, $\mathbb{R}$–trees, Bruhat–Tits buildings, etc.

Let us consider a geodesic space (X, ρ) and let, for $\kappa \in \mathbb{R}$, M^2_κ be the complete, simply connected, 2–dimensional Riemannian manifold of constant sectional curvature κ. Moreover, let D_κ be the diameter of M^2_κ which equals $\pi/\sqrt{\kappa}$ for $\kappa > 0$ and ∞ otherwise. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in X consists of the points $\{x_i\}$ and three geodesics joining them. If additionally $\rho(x_1, x_2) + \rho(x_2, x_3) + \rho(x_3, x_1) \leq 2D_\kappa$ then there is a unique up to isometries triangle $\Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in M^2_κ with $\rho(x_i, x_j) = d_\kappa(\bar{x}_i, \bar{x}_j)$. A geodesic triangle $\Delta(x_1, x_2, x_3)$ satisfies the $\mathrm{CAT}(\kappa)$ inequality if for each pair of points $p, q \in \Delta(x_1, x_2, x_3)$ and their comparison points $\bar{p}, \bar{q} \in \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ the following condition holds

$$\rho(p, q) \leq d_\kappa(\bar{p}, \bar{q}).$$

We say that X is a $\mathrm{CAT}(\kappa)$ space if each pair of points $x, y \in X$ with $\rho(x, y) \leq D_\kappa$ are joined by a geodesic and each triangle with perimeter smaller than $2D_\kappa$ satisfies the $\mathrm{CAT}(\kappa)$ inequality.

If each triangle satisfies the opposite inequality the space is called of curvature bounded below (in the Alexandrov sense) by κ. Clearly, there exist spaces which satisfy both inequalities, i.e., which at the same time have curvature bounded below and above. The easiest example is a Hilbert space with curvature constantly equal to 0. Another very well–known and widely studied example is the complex Hilbert ball with the hyperbolic metric (see [22] and, among others, [25], [40] and [46]). Let $\mathcal{B}$ be a unit open ball of a complex Hilbert space H equipped with the metric given by the formula

$$\rho(x,y) = \operatorname{arctanh}(1 - \sigma(x,y))^{1/2},$$

where

$$\sigma(x,y) = \frac{(1 - \|x\|^2)(1 - \|y\|^2)}{|1 - (x,y)|^2}$$

(see [22, p. 99]). Then $(\mathcal{B}, \rho)$ is called the complex Hilbert ball (with the hyperbolic metric). If one considers the orthonormal base $\{e_\lambda\}_{\lambda \in \Lambda}$ of H and a subset $\mathcal{D}$ of $\mathcal{B}$ such that

$$z \in \mathcal{D} \Leftrightarrow (z, e_\lambda) \in \mathbb{R} \qquad \text{for all } \lambda \in \Lambda,$$

then $\mathcal{D}$ with the same metric is called the real Hilbert ball (with hyperbolic metric) (see [22, Section II.32]). In the special case of $H = \ell^2$, the space $\mathcal{D}$ is denoted by $\mathbb{H}^\infty$ to emphasize the fact that it is isometric to the infinite dimensional Klein model of the hyperbolic space (see [P01, p. 644]).

Another very well-known example of a CAT(κ) space (for all $\kappa \in \mathbb{R}$) are $\mathbb{R}$-trees, i.e., uniquely geodesic spaces such that each geodesic triangle $\Delta(x,y,z)$ forms a tripod. The topic of $\mathbb{R}$-trees was deeply discussed, among others, in [11], [29], [P02] and [P10], while the notion of tripod can be found in [14, p. 2]).

Both examples mentioned above, the Hilbert balls and $\mathbb{R}$–trees, are also hyperbolic spaces in the sense of Gromov. We will come back to this notion in a moment but first we collect the main geometric properties of CAT(κ) spaces in the following proposition. Let us begin with the notion of Alexandrov angle in a geodesic space. Let c, c' be two geodesics issuing from a common point x and let $y = c(s)$, $z = c'(t)$. Then the angle $\angle_x(y,z)$ is defined by

$$\angle_x(y,z) = \limsup_{\substack{s',t' \to 0^+ \\ s'<s, t'<t}} \angle_{\bar{x}}(\bar{c}(s'), \bar{c}'(t')), \tag{4.1}$$

where $\angle_{\bar{x}}(\bar{c}(s'), \bar{c}'(t'))$ is the angle of the comparison triangle $\Delta(\bar{x}, \bar{c}(s'), \bar{c}'(t'))$ (see for instance [11, Definition I.1.12]).

Proposition 4.2. *Let X be a complete CAT(κ) space. Then:*

(i) the angles in the geodesic triangle $\Delta(x,y,z)$ are not greater than angles of the comparison triangle $\Delta(\bar{x}, \bar{y}, \bar{z})$ on M^2_κ;

(ii) X *is also a CAT(κ') space for all* $\kappa' > \kappa$*;*

(iii) under an additional assumption that $\kappa \leq 0$ *for each nonempty closed convex subset* C *there is a well defined, single-value and nonexpansive projection* $P_c \colon X \to C$ *defined by*

$$P_c(x) = \{y \in C : d(x, y) = d(x, C)\}.$$

These results can be found for instance in [11, II.1].

We say that a CAT(κ) space has the extension property if each geodesic $c \colon [0, l] \to X$ can be extended to a geodesic $c' \colon [0, \infty) \to X$ in such a way that $c'|_{[0,l]} = c$ (see [P02, p. 137]). Let us emphasize that this property can be introduced in a slightly different way in each geodesic space (see [11, Definition II.5.7]). However, in the case of mentioned before complete CAT(0) spaces these definitions coincide (see [11, Lemma II.5.8]).

All $\mathbb{R}$-trees, the Hilbert balls with hyperbolic metric, as well as all CAT(κ) spaces with negative κ are also hyperbolic in the Gromov sense. This means that for a uniquely geodesic space X there exists a nonnegative number δ such that in each geodesic triangle $\Delta(x_1, x_2, x_3)$ the following holds

$$d(x, [x_i, x_j] \cup [x_j, x_k]) \leq \delta \qquad \text{for all } x \in [x_i, x_k],$$

where $i, j, k \in \{1, 2, 3\}$ (see [11, Section III.H.1] and [14, Section 1.2]). Such a space is said to be δ-hyperbolic or hyperbolic in the sense of Gromov. Note that hyperbolicity can be also introduced in each metric space independently of the existence of geodesics and if X a uniquely geodesic space this definition is equivalent to the above one (see [13, Section 8.4.1] and [24]).

CAT(κ) spaces (with diameter smaller than $D_\kappa/2$) are also uniformly convex geodesic spaces (for the formal definition see for instance [P13, 35]), so each bounded sequence (x_n) of X has a unique asymptotic center, usually denoted by $A((x_n))$. Let us recall that $A((x_n))$ is a point for which

$$\limsup_n \rho(A((x_n)), x_n) = \inf_{x \in X} \limsup_n \rho(x, x_n).$$

In general, geodesic spaces do not have a unique asymptotic center for each bounded sequence, so spaces which satisfy this condition are said to have the unique asymptotic center property (see [P04]).

Now let us introduce a more general class of spaces than CAT(κ) ones. We say that a geodesic space is Busemann convex if for each pair of geodesics c, c' issuing from one point the following inequality holds

$$\rho(c(st), c'(s\tau)) \leq s\rho(c(t), c'(\tau)) \tag{4.2}$$

for each $s \in (0,1)$ (see [50, Chapter 8]). Clearly, each strictly convex Banach space is Busemann convex. Moreover, each Busemann convex space is uniquely geodesic but not vice versa. For example, one may consider a convex subset C of the 2–dimensional unit sphere S^2 with diameter smaller than $\pi/2$. Then C is not only uniquely geodesic but also satisfies the CAT(1) inequality. At the same time C is not Busemann convex. Here it is worth emphasizing that CAT(κ) spaces are Busemann convex for all nonpositive κ.

Let us notice that sometimes instead of Busemann convexity the authors use the notion of W–hyperbolic space (see [34]). In this case we do not assume that X is necessarily uniquely geodesic. Instead one may choose a family of geodesics $\mathcal{W}$ such that each couple of points is joined by a unique element of $\mathcal{W}$ and $\mathcal{W}$ satisfies (4.2).

A very fruitful method of building new geodesic spaces is the gluing operation. If $(X_\lambda, d_\lambda)_{\lambda\in\Lambda}$ is a family of metric spaces with closed subspaces $A_\lambda \subset X_\lambda$ and A is a metric space such that for each $\lambda \in \Lambda$ we have an isometry $i_\lambda \colon A \to A_\lambda$ then one may consider the quotient X of the disjoint union $\coprod_\Lambda X_\lambda$ by the equivalence relation generated by $[i_\lambda(a) \sim i_{\lambda'}(a) \forall a \in A, \lambda, \lambda' \in \Lambda]$. If we identify each X_λ with its image in X and write

$$X = \bigsqcup_A X_\lambda,$$

then X is called the gluing (or amalgamation) of the X_λ along A (see [11, Definition I.5.23]). Let the distance between $x \in X_\lambda$ and $y \in X_{\lambda'}$ be given by the formula:

$$\begin{aligned} d(x,y) &= d_\lambda(x,y) \qquad \text{if } \lambda = \lambda' \\ d(x,y) &= \inf_{a\in A}\{d_\lambda(x, i_\lambda(a)) + d_{\lambda'}(i_{\lambda'}(a), y)\} \qquad \text{if } \lambda \neq \lambda'. \end{aligned}$$

Then d is a metric on X (see [11, Lemma I.5.24]).

In the case of gluing CAT(κ) spaces the Retchenayk gluing theorem is a key result.

Theorem 4.3. *([11] Theorem II.11.3)*
Let $(X_\lambda, d_\lambda)_{\lambda\in\Lambda}$ be a family of CAT(κ) spaces with closed subspaces $A_\lambda \subset X_\lambda$. Let A be a metric space and suppose that for each $\lambda \in \Lambda$ we have an isometry $i_\lambda \colon A \to A_\lambda$. Let $X = \bigsqcup_A X_\lambda$ be the space obtained by gluing the spaces X_λ along A. If A is a complete CAT(κ) space (in which case each A_λ is D_κ –convex and complete), then X is a CAT(κ) space.

Now let us remark that the gluing of Busemann convex spaces does not necessary give a space of the same type. A counterexample in this sense as well as additional conditions under which the gluing is still a Busemann convex space can be found in [P26].

At the end of this section let us focus on the definition of the boundary of a space. There exist various methods of defining the boundary at infinity for a geodesic space. In the case of the Busemann convex space, the most adequate one is via geodesic rays. Moreover, this definition is directly related to the geodesic boundedness of the space, which play a key-role in the main results.

Let X be a geodesic space. We say that two geodesic rays $c, c' \colon [0,\infty) \to X$ are asymptotic if there exists a positive number such that $d(c(t), c'(t)) \leq M$ for all $t > 0$ (see [11, Definition II.8.1]). Then the geodesic boundary $\partial^g X$ is defined as the set of equivalence classes of geodesic rays, where two rays are equivalent if they are asymptotic. If X is a Busemann convex space, the extended space $X \cup \partial^g X$ can be equipped with the cone topology, which coincides with the natural topology on X. For a point at infinity $\xi \in \partial^g X$ this is a compact-open topology on geodesic segments and geodesic rays issuing from a fixed base point $o \in X$ (for a more precise definition see [P03, Section 2]). In [15, Lemma 5.3] it was proved that this definition is independent of the choice of a base point so is well defined. For some special subclasses of Busemann convex spaces the above definition coincides with the boundary at infinity in the sense of Gromov or with the Kuratowski compactification of X via Busemann functions (see for instance [1], [8] and [27]). In particular, for complete CAT(0) spaces the following result holds:

Proposition 4.4. *([8] Proposition II.2.5)*
Let (x_n) be a sequence in X with $\rho(x_0, x_n) \to \infty$. Then $b(x_n, x_0, \cdot) = \rho(x_n, \cdot) - \rho(x_n, x_0)$ converges to a Busemann function f if and only if $[x_0, x_n]$ tends to a geodesic ray $\sigma = \sigma_{x_0,\xi}$. Furthermore, we have $f = b_\sigma$, where

$$b_\sigma(x) := \lim_{t\to\infty} (\rho(\sigma(t), x) - t).$$

If X is a complete CAT(κ) space with negative κ then all mentioned boundaries coincide (see [10] and [21]).

4.3 The state of the art before my research

Since my main results are related to the fixed point property for non-expansive mappings defined on unbounded or non-convex CAT(0) spaces I would like to focus for a moment on the results obtained by a wide group of mathematicians at the turn of the twentieth and twenty-first century in this concrete direction. Here we should notice that in the case of unbounded CAT(0) spaces so far all results have been devoted to a few very concrete and specific subclasses of such spaces. As a consequence, they have not clarified the full picture of the situation. Moreover, the methods applied there could not be used in more general cases. Let us begin our consideration by collecting these results.

Since each Hilbert space is a CAT(0) space, it is a natural way to begin with a very well-known result proved in 1980 by W.O. Ray:

Theorem 4.5. *([51] Theorem 1)*
Let K be a closed and convex subset of a real Hilbert space H. Then K has the fixed point property for nonexpansive mappings if and only if K is bounded.

Seven years later the same theorem was proved by Sine in a much easier way (see [57]). Although the new proof was based on the Banach – Steinhaus principle it cannot be applied to a more general subclass of Banach spaces. The stronger version of this result for Hilbert spaces was proved by F. Kohsaka in [38]. The last result holds also in each smooth, strictly convex, reflexive Banach space but only for the case of mappings of firmly-nonexpansive type, i.e., mappings satisfying the following condition

$$(Tx - Ty, JTx, JTy) \leq (Tx - Ty, Jx - Jy), \tag{4.3}$$

where J is a normalized duality mapping. Obtaining a counterpart of Ray's theorem for nonexpansive mappings is still an open problem.

Here it is worth emphasizing one thing. While the existence of fixed points for nonexpansive mappings defined on specific spaces is still an open problem, the characteristic of sets with the almost fixed point property is well-known so we do not focus on it in the main part of this report. Let us recall that X has the almost fixed point property (for nonexpansive mapping) if for each nonexpansive $T\colon C \to C$ defined on nonempty, closed and convex $C \subset X$,

$$\inf_{x \in C} \rho(x, Tx) = 0.$$

In [56] I. Shafrir proved the following fact:

Theorem 4.6. *(compare to [56, Theorem 2.4])*
Let X be a Busemann convex space with the extension property. Then a convex $C \subset X$ has the almost fixed point property if and only if C does not contain directional curves.

A curve $\gamma\colon [0, \infty) \to X$ is said to be directional if there is $b \geq 0$ such that

$$t - s - b \leq \rho(\gamma(s), \gamma(t)) \leq t - s$$

for all $t \geq s \geq O$. The known results of Ray (for Hilbert spaces) and Goebel and Reich (in the Hilbert balls) show that the existence of fixed points does not follow directly from the almost fixed point property and so the above result due to Shafrir does not have a deeper application to our problem. At the end let us note that sometimes (for instance in the mentioned paper of Shafrir) this property is called the approximate fixed point property but here we use a more widespread notion (see for instance [52]).

A complete different situation to the one described above takes place if we consider the complex Hilbert ball with the hyperbolic metric. Then as it was shown by K. Goebel and S. Reich for each nonexpansive mapping defined on $\mathcal{B}$ we have only one of two possibilities:

Theorem 4.7. *(see [22] Theorem 24.1 and 25.2)*
Let $T\colon \mathcal{B} \to \mathcal{B}$ be a nonexpansive mapping. Then precisely one of the following is true:

(i) T has a nonempty, closed and convex set of fixed point and each approximating curve of first kind $z_t(a)$ tends to the projection of a onto $Fix(T)$ (see (4.5)*);*

(ii) there is precisely one "sink point" on the boundary of $\mathcal{B}$ such that each approximating curves of first kind tends to this point with respect to norm.

If we consider now the more general case of all nonexpansive mappings defined on nonempty, closed and convex subset K of $\mathcal{B}$ then from (iii) of Proposition 4.2 we get that for each fixed point free mapping $T\colon K \to K$ there exists a point at the boundary which belongs to $\bar{K}$ (the closure of K with respect to norm). So we finally obtain a full characterization of subsets of $\mathcal{B}$:

Corollary 4.8. *([P03] Corollary 4.4)*
Let a nonempty $K \subset \mathcal{B}$ be convex and closed. Then K has the fixed point property for nonexpansive mappings if and only if K is geodesically bounded.

Let us recall that a geodesic space is said to be geodesically bounded if it does not contain a geodesic ray.

We have a similar situation in the case of $\mathbb{R}$-tree, i.e., spaces which can be treated as CAT$(-\infty)$ spaces. At the turn of the century the following was proved by W. A. Kirk:

Theorem 4.9. *([30] Theorem 3.4.)*
Suppose that X is a geodesically bounded complete $\mathbb{R}$-tree. Then every continuous mapping $f\colon X \to X$ has a fixed point.

Clearly, the above theorem gives a full characterization of sets with the fixed point property for continuous mappings. Indeed, again on account of (iii) of Proposition 4.2 we know that the projection onto a geodesic ray is a nonexpansive mapping, so considering the movement along this ray we obtain the existence of a fixed point free nonexpansive and so continuous mapping. Finally this leads to the same full characterization of sets as for the complex Hilbert ball.

Corollary 4.10. *Let X be the complete $\mathbb{R}$-tree and a nonempty $K \subset X$ be convex and closed. Then K has the fixed point property for nonexpansive mappings if and only if K is geodesically bounded.*

Partially this result was proved by the same author in co-operation with R. Espínola for the commuting family of nonexpansive mappings (see [20, Theorem 4.3]).

Let us notice that all examples described above can be treated as very special subclasses of CAT(0) spaces and the tools using there cannot be modified in order to be applied in more general cases. So the main idea of my research in this direction initiated by fruitful discussions with Prof. Rafael Espínola from the University of Seville was to clarify and deepen the knowledge about the fixed point property for unbounded CAT(0) spaces.

Now let us consider a nonexpansive mapping defined on a non-convex domain. Similarly as it was done in the previous part of this section we begin with the very natural case of Hilbert spaces. Let us assume that D is a nonempty subset of a Hilbert space H. The first result devoted to the existence of fixed points for a nonexpansive mapping $T\colon D \to D$ was proved by K. Goebel and R. Schöneberg in the following form:

Theorem 4.11. *(see [23, Theorem] and [55, Corollary 3.2])*
Let T be an nonexpansive self-mapping of a non-empty subset D of H. Then T has a fixed point in D if and only if $(T^n x)$ is bounded for some (hence for all) $x \in D$ and for any $y \in \bar{co}\{T^n x : n > 0\}$ there is a unique $p \in D$ such that $\|y - p\| = \inf_{z\in D} \|y - z\|$.

Let us recall that in a geodesic space we define the convex hull inductively. Let $Y \subset X$, we denote by $G^1(Y)$ the union of all geodesic segments in X with endpoints in Y. For each $n > 1$ let

$$G^n(Y) := G^1\left(G^{n-1}(Y)\right).$$

Then the convex hull of Y is equal to

$$co\, Y = \bigcup_{n>0} G^n(Y).$$

By $\bar{co}\, Y$ we denote the closure of this set (see [P02, pp. 137–138]).

Precisely the same result but for more general mappings, namely asymptotically nonexpansive in the intermediate sense, was proved by B. D. Rouhani in 2002 (see [55, Theorem 3.1]). Let us recall the definitions of various types of asymptotically nonexpansive mappings. We recall only notions which are used in the sequel.

Definition 4.12. ([55] p.1099)
Let D be a non-empty subset of a metric space (X, ρ). A mapping $T\colon D \to D$ is said to be of asymptotically non-expansive type if for each $x \in X$:

$$\limsup_{n\to\infty} M(x,n) \leq 0,$$

where $M(x,n) = \sup_{y\in X} \left(\rho(T^n x, T^n y) - \rho(x,y)\right)$. If the numbers $M(x,n)$ can be commonly bounded by a sequence tending to 0 when $n \to 0$ we say that T is asymptotically nonexpansive in the intermediate sense.

In the proof of his result Rouhani applied a constructive method which played a key–role in his earlier paper devoted to the problem of extending of nonexpansive mappings. More precisely, in [54] B.D. Rouhani gave a constructive proof of the following version of the Kirszbraun–Valentine theorem:

Theorem 4.13. *([54] Theorem 3.2)*
Let D be a nonempty subset of a real Hilbert space H, and $T\colon D \to D$ a nonexpansive mapping. Then T has an absolute fixed point in H if and only if the sequence $(T^n x)$ is bounded for some $x \in D$ (and hence for all $x \in D$). In this case, for any $z \in D$, the asymptotic center of $(T^n z)$ is an absolute fixed point for T. Moreover the mapping U from D to the set AF of absolute fixed points of T, corresponding to each $z \in D$ the asymptotic center of $(T^n z)$ is nonexpansive.

First let us recall the definition of an absolute fixed point which appeared in the previous theorem.

Definition 4.14. ([P02] Definition 2.12, [54] Definition 2.2)
Let X be a metric space, $D \subset X$ nonempty and $T\colon D \to D$ a nonexpansive mapping. We say that $x \in X$ is an absolute fixed point of T if the extension $\tilde{T}\colon D \cup \{x\} \to D \cup \{x\}$ such that $\tilde{T}x = x$ is nonexpansive and if x is a fixed point for any nonexpansive extension of T to the union of D and a subset of X containing x.

Now one may look at a Hilbert space as a space with constant curvature (having the same upper and lower curvature bound in the sense of Alexandrov). So this leads to the natural question how a nonexpansive (or nonexpansive type) mapping defined on a subset of a space of constant curvature behaves. The first positive result devoted to the extension of such mappings was proved by T. Kuczumow and A. Stachura in the case of a one-dimensional complex Hilbert ball (with the hyperbolic metric) and the real Hilbert ball $\mathcal{D}$ (see [41] and [42]). Next these results were generalized by U. Lang and V. Schroeder in the following form:

Theorem 4.15. *([43] Theorem A.)*
Let $\kappa \in \mathbb{R}$, and let X, Y be two geodesic metric spaces such that all triangles of perimeter $< 2D_\kappa$ in X or Y are κ-thick or κ-thin respectively. Assume that Y is complete. Let S be an arbitrary subset of X and $f\colon S \to Y$ a 1-Lipschitz map with diam$(S) \leq D_\kappa/2$. Then there exists a 1-Lipschitz extension $\bar{f}\colon X \to Y$ of f.

The concept of κ-thick i κ-thin spaces used in Theorem 4.15 mean the lower or upper local curvature bound by κ respectively.

In the above mentioned paper of Kuczumow and Stachura, there is an example which shows that in the complex Hilbert ball $\mathcal{B}$ one may find a nonexpansive mapping actually an isometry defined on subsets D of $\mathcal{B}$ which cannot be extended to the whole $\mathcal{B}$ (see [22, Remark, p. 119] for an example in the finite dimensional case and [42, Example 1] for an example in the infinite dimensional one). Since the two-dimensional space $\mathcal{B}$ has curvature bounded above and below by two different numbers, one may expect further research on the existence of fixed points or absolute fixed points to go in the direction of mappings defined on spaces with constant curvature as it was done in my paper [P02] written in cooperation with Rafael Espínola.

4.4 Unbounded CAT(0) spaces and nonexpansive mappings

As mentioned before, the results devoted to the behavior of nonexpansive mappings on concrete examples of CAT(0) spaces as well as Ray's theorem motivated me to study the problem of existence of fixed points for nonexpansive mappings defined on unbounded subsets of geodesic spaces.

At the beginning of our research with R. Espínola we focused on more general examples of CAT(0) spaces for which the fixed point property holds or does not hold. First we proposed the following construction (based on the gluing of CAT(0) spaces – see Theorem 4.3) of a wide class of spaces which have fixed points for nonexpansive mappings.

Proposition 4.16. *([P01] Proposition 3.1)*
Let C be a complete CAT(0) space which can be written as

$$C = C_0 \cup \bigcup_{n=1}^{\infty} C_n,$$

where:

(i) C_0 *is bounded closed and convex,*

(ii) C_n *is closed and bounded with* $C_0 \cup C_n$ *convex for any* n*,*

(iii) $\{C_n\}$ *is a family of pairwise disjoint sets such that diam*(C_n) *tends to infinity as* $n \to \infty$*,*

(iv) $W_n = C_0 \cap C_n$ *is nonempty, diam*$(W_n) \leq \alpha$ *for each* n *and dist*$(W_n, W_m) = inf\{d(x, y) : x \in W_n, y \in W_m\} \geq \alpha$ *for a certain* $\alpha \geq 0$ *and any* $n \neq m$*.*

Then, C *is geodesically bounded and unbounded, and has the fixed point property.*

The proof of this proposition was based on the characterization of the almost fixed point property for nonexpansive mappings originally due to I. Shafrir (see Section 4.3) and in the settings of CAT(0) spaces given by W. A. Kirk in the following form:

Theorem 4.17. *([31] Theorem 25)*
A closed and convex subset of a complete CAT(0) spaces with the extension property has the almost fixed point property for nonexpansive mappings if and only if it does not contain a geodesic ray.

An immediate consequence of Proposition 4.16 is a counterpart of the Kirk's result (see Theorem 4.9) for nonexpansive mappings.

Corollary 4.18. *([P01] Corollary 3.4)*
Let X *be an unbounded but geodesically bounded complete* $\mathbb{R}$*–tree, then it has the fixed point property for nonexpansive mappings.*

Another example of a space satisfying the assumptions of Proposition 4.16 is the following one:

Example 4.19. ([P01] Example 3.3)
Let C be the closed unit ball in ℓ_2 and $\{e_n\}$ the elements of its standard basis. For each $n \in N$ let us consider

$$C_n = \bar{co}\left(B\left(\left(1 - \frac{1}{n}\right)e_n, \frac{1}{n}\right) \cup \{ne_n\}\right),$$

where the $\bar{co}$ stands for the closed and convex hull. Now take $N \in \mathbb{N}$ so that $\text{dist}(C_n, C_m) = inf\{d(x, y) : x \in C_n, y \in C_m\} > 1$ for any $n, m \geq N$. Consider now the gluing of C with C_n for $n \geq N$. By the Retchenayk gluing theorem (see Theorem 4.3), these gluings are all CAT(0) spaces and, gluing again all of them,

$$X = C \cup \bigcup_{n \geq N} C_n$$

is a complete CAT(0) space.

Further we introduced the concept of the property U. This property permits to separate CAT(0) spaces that have the fixed point property only on bounded subsets from those that have the fixed point property also for unbounded but geodesically bounded subsets as is the case of the complex Hilbert ball with the hyperbolic metric. Clearly, we only consider closed and convex subsets of complete spaces.

Definition 4.20. ([P01] Definition 4.1)
Let X be an unbounded geodesic space. Then we say that X has the property of the far unbounded convex set (property U, for short) if for any convex closed and unbounded subset Y of X either Y is geodesically unbounded or for each closed convex and unbounded $K \subset Y$ and $x \in K$ there exists a closed convex and unbounded subset K_1 of K such that $d(x, K_1) \geq 1$.

Natural examples of spaces which satisfy this property are the following ones:

Proposition 4.21. *([P01] Proposition 4.2)*
Let X be a reflexive Banach space. Then X has property U.

Proposition 4.22. *([P01] Proposition 4.5)*
Any unbounded locally compact complete CAT(0) space has property U.

Let us notice that in the case of locally compact CAT(0) spaces we may obtain much more as Theorem 3.6 in [P01] shows, compare also with results from Section 5.2.

At the same time let us notice that:

Proposition 4.23. *([P01] Proposition 4.3)*
If an unbounded $\mathbb{R}$–tree is geodesically bounded then it does not have property U.

Using this property one may state that:

Theorem 4.24. *([P01] Theorem 5.1)*
Let X be a complete CAT(0) space. Suppose also that X satisfies property U. Then a nonempty closed convex subset $Y \subset X$ has the fixed point property for nonexpansive mappings if and only if Y is bounded.

So as a natural consequence of the above fact and Proposition 4.21 we obtain an alternative proof of Ray's result. At the same time let us notice that in the above theorem we suppose that X is a CAT(0) space, so we cannot deduce from it any information about fixed point property for any Banach spaces which are not inner product ones.

As mentioned before all CAT(κ) spaces with $\kappa < 0$ are hyperbolic in the sense of Gromov. This fact leads to the following conclusion:

Proposition 4.25. *([P01] Corollary 5.5.)*
Let X be a CAT(κ) space with $\kappa < 0$ and let x_0, x and $y \in X$ such that there exists r, $\varepsilon > 0$ with $\rho(u, v) \geq \varepsilon$, where u and v are, respectively, the metric projection of x and y onto $B(x_0, r)$, then there exists $R > 0$, depending only on r and ε, such that

$$B(x_0, R) \cap [x, y] \neq \emptyset.$$

Using this fact we showed that:

Theorem 4.26. *([P01] Theorem 5.6)*
Let X be an unbounded and complete CAT(κ) space, $\kappa < 0$, containing unbounded but geodesically bounded subsets, then X fails property U.

The results contained in [P01] allowed us to raise the following questions devoted to the structure of unbounded spaces with the fixed point property which motivated my further research on this topic.

Remark 4.27. ([P01] Remark 5.7)

(i) Example 4.19 shows that it is not necessary for a CAT(0) space to be a CAT(κ) space for some $\kappa < 0$ in order to fail Ray's theorem. Still the space provided by Example 4.19 is δ-hyperbolic. Therefore one step farther in the above problem is to consider whether Ray's theorem fails on any non locally compact and complete CAT(0) space which is δ–hyperbolic for some $\delta \geq 0$. Or, even more, on any Busemann convex and δ–hyperbolic non locally compact and complete geodesic space.

(ii) Any nonexpansive self-mapping defined on an unbounded closed and convex subset of an $\mathbb{R}$–tree, the Hilbert ball or a space of constant negative curvature has a fixed point if and only if it is geodesically bounded. May this same result be obtained for CAT(κ) spaces with $\kappa < 0$?

(iii) Is property U a necessary condition for a CAT(0) space to satisfy Ray's theorem?

4.5 The fixed point property in unbounded sets

According to the question (ii) from Remark 4.27 in the next step of my research I focused on spaces with curvature bounded above by a negative numbers, i.e., CAT(κ) spaces with negative κ. In the paper [P03] I gave the following positive answer to the question from [P01]:

Theorem 4.28. *([P03] Theorem 4.1)*
Let X be a complete CAT(-1) space and let a nonempty $K \subset X$ be closed and convex. Then K has the fixed point property for nonexpansive mappings $T\colon K \to K$ if and only if K is geodesically bounded.

The assumption that the curvature is bounded precisely by -1 played only a technical role so it could be dropped as the following corollary shows:

Corollary 4.29. *([P03] Corollary 4.2)*
Let X be a complete CAT(κ) space with $\kappa < 0$ and a nonempty $K \subset X$ be closed and convex. Then K has the fixed point property for nonexpansive mappings $T\colon K \to K$ if and only if K is geodesically bounded.

This result is a natural generalization of the following theorem due to K. Goebel and S. Reich about the behavior of mappings defined on the very specific case of the real Hilbert ball:

Theorem 4.30. *([22] Theorem 32.3)*
A closed and convex subset C of $\mathcal{D}$ (the real Hilbert ball with the hyperbolic metric) has the fixed point property (and the almost fixed point property) for nonexpansive mappings if and only if it is geodesically bounded.

Let us notice that in contrast to the case of the fixed point property for continuous mappings (see Section 5.2) I did not need to assume that the spaces also have curvature bounded below. An example of such space is the complex (or real) Hilbert ball with the hyperbolic metric, where the curvature (in the sense of Alexandrov) is bounded above by -1 and below by -4 (this fact was precisely explained in [P03, pp. 332–333]).

After these digressions, let us go back to the discussion on the main result from [P03]. The proof of this theorem is based very deeply on the geometry of CAT(κ) spaces. One geometrical fact proved in [P03] – namely, the estimation of an edge of the geodesic triangle holds only in the case of spaces with negative curvature as the following lemma shows:

Lemma 4.31. *([P03] Lemma 3.2)*
Let X be a CAT(-1) space and consider a sequence of triangles $(\Delta(x_n, y_n, z_n))_{n=1}^{\infty}$ such that

$$\angle_{y_n}(x_n, z_n) \geq \frac{\pi}{2}, \qquad n \in \mathbb{N},$$

and

$$d(x_n, y_n) \to \infty, \qquad d(x_n, z_n) - d(x_n, y_n) \to 0 \qquad \textit{for } n \to \infty.$$

Then $d(y_n, z_n) \to 0$.

Since the last lemma works only in CAT(κ) spaces when the parameter κ is negative so the reasoning method from [P03] cannot be directly applied in the more general case of hyperbolic (in the sense of Gromov) CAT(0) spaces. Let us remind that the examples of CAT(0) spaces which are also hyperbolic in the Gromov sense are very natural and one has already been proposed in Example 4.19.

The result on the existence of fixed points for nonexpansive mappings defined on geodesically bounded complete CAT(0) spaces which are also hyperbolic in the sense of Gromov was shown in [P04]. As it was already mentioned Lemma 4.31 holds only in CAT(κ) spaces with $\kappa < 0$, so the proof of this result is different from the one given in [P03]. During my research on this topic I noticed that we could drop the CAT(0) condition and assume instead two weaker properties of X. More precisely, we need the Busemann convexity which gives us the estimation of distances between points on two geodesics issuing from a common point and the asymptotic center. Since it is well-known which Banach spaces satisfy the unique asymptotic center property our considerations lead to the generalization of Example 4.19 for all subsets of the same form but in ℓ_p spaces (for all $p \in (1, \infty)$). Thus, the main result of [P04] reads as follows:

Theorem 4.32. *([P04] Theorem 3.1)*
Let X be a complete Busemann space with the unique asymptotic center property. Moreover, let us suppose that X is δ-hyperbolic for some nonnegative δ. If C is a convex and closed subset of X, then the following facts are equivalent:

a) C is geodesically bounded.

b) Each nonexpansive mapping $T\colon C \to C$ has at least one fixed point.

c) Each firmly nonexpansive mapping $T\colon C \to C$ has at least one fixed point.

Let us notice that in this characterization we also consider the concept of firmly nonexpansive mappings, i.e., in the case of geodesic spaces the mapping satisfying the following inequality

$$\rho(Tx, Ty) \leq \rho((1-\lambda)x + \lambda Tx, (1-\lambda)y + \lambda Ty)$$

(see also condition (4.3)). In [22, p. 124] the authors showed the relation between nonexpansive and firmly nonexpansive mappings defined on the complex Hilbert ball $\mathcal{B}$. More precisely, if $T\colon \mathcal{B} \to \mathcal{B}$ is nonexpansive then one may define a mapping $U_t\colon \mathcal{B} \to \mathcal{B}$ where $U_t(x)$ is the unique fixed point of the contraction

$$G_t(y) := (1-t)Ty + tx, \tag{4.4}$$

i.e., $U_t(x) = (1-t)TU_t(x) + tx$. Then the mapping U_t is firmly nonexpansive and has the same fixed points as T. Very recently, in [3, Proposition 3.2] it was shown that the same construction works also in the class of Busemann convex spaces which gives us directly the equivalence of conditions *b)* and *c)*.

A main role in the proof of Theorem 4.32 is played by the behavior of the approximating curves defined in the following way:

Definition 4.33. Let X be a complete Busemann convex space and $T\colon X \to X$ a nonexpansive mapping. Let us fix $x \in X$. Then for each $t \in (0,1]$ the mapping G_t defined in (4.4) is a contraction, so it has a unique fixed point. Let us denoted it by z_t. Then the set

$$\{z_t : \ t \in (0,1]\} \tag{4.5}$$

is said to be the approximating curve (of the first kind) issuing from x.

Motivated by the results on nonexpansive mappings defined on the real and complex Hilbert balls with the hyperbolic metric due to K. Goebel and S. Reich (see Section 25 in [22]) I also studied in more detail how the approximating curves defined by (4.5) behave in the general case of Busemann spaces. Clearly (supposing in our case that the space has the unique asymptotic center property), if there is a bounded sequence (z_{t_n}) with $t_n \to 0^+$ then the mapping T must have a fixed point. So it is worth paying attention for the case of the fixed point free nonexpansive mappings. Then one may prove that:

Theorem 4.34. *([P04] Theorem 4.1)*
Let X be a complete Busemann space with the unique asymptotic center property. Moreover, let us assume that X is δ-hyperbolic for some positive δ.
If $T\colon C \to C$ is a fixed point free nonexpansive mapping, where C is a convex and closed subset of X, then there exists a point $\xi \in \partial^g X$ such that for each $x \in C$ the approximating curve issuing from x and defined by (4.5) *converges to ξ with respect to the cone topology.*

This result is a natural generalization of a very special case of the complex Hilbert ball (see *(ii)* in Theorem 4.7). Let us note that in this case the boundary at infinity (the set $\partial^g X$) is nonempty so we may consider mappings defined on $X \cup \partial^g X$ instead of mappings defined on X. This set is equipped with the cone topology which is not metrizable. So we may assume that the mapping $T\colon X \cup \partial^g X \to X \cup \partial^g X$ is continuous on its domain and nonexpansive on a subset of X. Then we get:

Corollary 4.35. *Let X be a complete Busemann space with the unique asymptotic center property. Moreover, let us suppose that X is δ-hyperbolic for some positive δ.*
If $T\colon X \cup \partial^g X \to X \cup \partial^g X$ is continuous with respect to the cone topology and there exists a nonempty closed and convex $C \subset X$ such that $T|_C\colon C \to C$ is nonexpansive, then T has at least one fixed point in $X \cup \partial^g X$.

4.6 Fixed point free nonexpansive mappings

If the mapping $T\colon X \to X$ is nonexpansive and fixed point free then we know that for each $x \in X$ the approximating curve issuing from x tends (with respect to the cone topology) to a point at infinity $\partial^g X$. As it follows from Theorem 4.34 for two different points x_1 and x_2 of X their curves must converge to the same point at infinity. So the next natural question here is how the orbit of T behaves in this case? In the sequel the orbit of T, i.e, the set $\{T^n x :\ n \in \mathbb{N}\}$ will be also called the Picard iterative sequence. My results on this topic were presented in the paper [P05]. More precisely, I proved that:

Theorem 4.36. *([P05] Theorem 4.2)*
Let X be a complete Busemann space with the unique asymptotic center property. Let us also assume that X is δ-hyperbolic for some positive δ and $T\colon X \to X$ is a fixed point free nonexpansive mapping. Then there is a point ξ at infinity such that for each $x \in X$ there is a subsequence of the Picard iterative sequence $(T^n x)$ which tends to ξ with respect to the cone topology.

Moreover, the point ξ is the limit for approximating curves.

Clearly, since the complex Hilbert ball $\mathcal{B}$ is a very special case of the class of spaces satisfying the assumptions of the above theorem there exist nonexpansive fixed point free mappings for which the orbit has a bounded subsequence. An example of such a mapping defined on $\mathcal{B}$ can be found in [59]. However, as also mentioned in [P05], in this situation each unbounded subsequence $(T^{k_n} x)$ of $(T^n x)$ which satisfies the following condition:

$$\forall M > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N :\ \rho(x, T^{k_n} x) \geq M, \tag{4.6}$$

must also converge with respect to the cone topology to the same point at infinity.

However, as one may expect there are also mappings for which the whole orbit is unbounded in the sense as in formula (4.6). This condition turns out to be closely related to the almost fixed point property:

Theorem 4.37. *([P05] Theorem 4.3)*
Let X be a complete Busemann space with the unique asymptotic center property. Let us also assume that X is δ-hyperbolic for some positive δ and $T\colon X \to X$ is a fixed point free nonexpansive mapping. If, additionally, the number $D = \inf_{x\in X} \rho(x, Tx)$ is positive then for each $x \in X$ the whole orbit $(T^n x)$ tends to the same point at infinity.

In the previous section we defined the notion of firmly nonexpansive mappings. Now we focus on mappings satisfying a more general definition due to R. Smarzewski [58].

Definition 4.38. ([P05] Definition 5.1)
Let X be a unique geodesic space and $\lambda \in (0, 1)$. A mapping $T\colon X \to X$ is said to λ–firmly nonexpansive if

$$\rho(Tx, Ty) \leq \rho(\lambda Tx + (1 - \lambda)x, \lambda Ty + (1 - \lambda)y),$$

for all $x, y \in X$.

In linear spaces the point $\lambda a+(1-\lambda)b$ is defined in a unique way. However, if X is supposed only to be a geodesic space we should consider all geodesics joining points a and b, so for the precise definition of $\lambda a + (1 - \lambda)b$ we assume that X is uniquely geodesic. Moreover, in the class of Banach spaces each λ–nonexpansive mapping is nonexpansive. The same property does not hold in each geodesic (even uniquely geodesic) space, but it is true if X is Busemann convex. In the case of geodesic and Busemann convex spaces λ–firmly nonexpansive mappings have called the interest recently (see for instance the papers [3] and [48]). All these results motivated me to consider the problem of convergence of the Picard iteration sequence also for this kind of mappings. I showed that:

Corollary 4.39. *([P05] Theorem 5.2)*
Let X be a complete Busemann space with the unique asymptotic center property. Moreover, let us assume that X is δ-hyperbolic for some positive δ. If $T\colon X \to X$ is a fixed point free λ-firmly nonexpansive mapping (for some $\lambda \in (0, 1)$), then for each $x \in X$ the orbit $(T^n x)$ converges to the same point at infinity.

This result follows from the fact that a λ–firmly nonexpansive mapping T is asymptotically regular at an arbitrary point x, i.e., it satisfies

$$\lim_n \rho(T^n x, T^{n+1} x) = 0.$$

Thus, any bounded subsequence of the orbit has a unique asymptotic center which must be a fixed point and contradicts our assumption. In the same way we may obtain the result for the case of averaged mappings. First let me recall this definition.

Definition 4.40. ([P05] Definition 5.2)
A mapping $U\colon X \to X$ defined on a uniquely geodesic space is called averaged if there is a nonexpansive mapping $T\colon X \to X$ and a number $c \in (0,1)$ such that

$$Ux = cTx + (1-c)x, \qquad x \in X,$$

This class was introduced in [7] for the case of Banach spaces. In the eighties they appeared first papers devoted to the averaged mappings defined on the complex Hilbert ball $\mathcal{B}$ (see [53]) and recently, properties of averaged mappings defined on uniquely geodesic spaces were studied for instance in [48]. In this case of spaces we can prove that:

Corollary 4.41. *([P05] Theorem 5.3)*
Let X be a complete Busemann space with the unique asymptotic center property. Moreover, let us assume that X is δ-hyperbolic for some positive δ. If $U\colon X \to X$ is a fixed point free averaged mapping, then for each $x \in X$ the orbit $(U^n x)$ converges to the same point at infinity.

A useful tool in the study of the behavior of a function at infinity are Busemann functions. In the case of CAT(0) spaces this notion is directly connected with the convergence with respect to the cone topology as shown in Proposition 4.4. If we do not assume that X is a CAT(0) space we cannot expect that the limit

$$\lim_n \left(\rho(x_n, \cdot) - \rho(x_n, x_0)\right),$$

where (x_n) tends to the point at infinity, exists. So in [P05] I considered functions of the type

$$b(x) = \limsup_n \left(\rho(z_{t_n}, x) - \rho(z_{t_n}, x_0)\right) \tag{4.7}$$

with (z_{t_n}) being the sequence of points of the approximating curve issuing from a fixed point. For functions of this type I showed that the horoballs B_M, $M \in \mathbb{R}$, are T–invariant. Let us assume that

$$B_M = \{x \in X\colon\ b(x) < -M\},$$

where b is the function defined in (4.7), with a fixed point $x_0 \in X$ and a sequence of points (z_{t_n}) of the approximating curve issuing from x_0.

Theorem 4.42. *([P05] Theorem 3.1)*
Let X be a complete Busemann space with the unique asymptotic center property. Moreover, let us assume that X is δ-hyperbolic for some positive δ. If $T\colon X \to X$ is a fixed point free nonexpansive mapping, then for each function defined in (4.7) *all its horoballs B_M are T–invariant.*

Finally, let us repeat that all results included in the section hold for (complete) CAT(κ) spaces with $\kappa < 0$ as well as for (complete) CAT(0) spaces, which are hyperbolic in the sense of Gromov.

4.7 Fixed points and absolute fixed points in non-convex subsets of CAT(0) spaces

We begin our considerations on mappings defined on non-convex subsets of geodesic spaces from the natural counterpart of Theorem 4.13 for spaces of constant non-positive curvature, i.e., spaces in which each triangle is isometric to the comparison triangle on the two-dimensional model space M_κ^2 (we suppose that κ is a nonpositive number). One may show that:

Theorem 4.43. *([P02] Theorem 3.6.)*
Let X be a complete space of constant curvature $\kappa \in (-\infty, 0]$, $D \subset X$ nonempty and $T\colon D \to D$ nonexpansive. Assume that $(T^n x)$ is bounded for some $x \in D$. Then $A((T^n x))$ is an absolute fixed point of T. Moreover, the sequence $(\rho(T^n y, A((T^m x))))_{n\in\mathbb{N}}$ is non increasing and the mapping U from D into the set of absolute fixed points of T given by $U(x) = A((T^n x))$ is nonexpansive.

The proof was based on, among others, the following observation which is a natural consequence of Theorem 4.15:

Theorem 4.44. *([P02] Theorem 3.4.)*
Let X be a complete space of constant curvature $\kappa \in (-\infty, 0]$. Let $D \subset X$ nonempty and $T\colon D \to X$ nonexpansive. Then, given $p \notin D$, there exists a nonexpansive extension of T to $D \cup \{p\}$.

So we only needed to show that the mapping U is nonexpansive. This part of the proof was based on some geometrical properties of model spaces. From the previous result one may additionally draw the following conclusions about the existence of fixed points:

Corollary 4.45. *([P02] Corollary 3.7.)*
Let X be a complete space of constant curvature $\kappa \in (-\infty, 0]$, $D \subset X$ nonempty and $T\colon D \to D$ nonexpansive. Then T has a fixed point if and only if there is $x \in D$ for which the sequence of its iterates $(T^n x)$ is bounded and there is a unique $y \in D$ such that

$$d(A((T^n x)), y) = d(A((T^n x)), D).$$

Corollary 4.46. *([P02] Corollary 3.8.)*
Let X be a complete space of constant curvature $\kappa \in (-\infty, 0]$, $D \subset X$ nonempty and $T\colon D \to D$ nonexpansive. Then T has a fixed point in D if and only if there is $x \in D$ for which the sequence of its iterates $(T^n x)$ is bounded and for any $y \in \bar{co}\{T^n x : n > 0\}$ there is a unique $p \in D$ such that $d(y, D) = d(y, p)$.

In the case of asymptotically nonexpansive mappings we did not expect the same results as for nonexpansive mappings to hold true. We were rather interested in taking a look at these mappings and showing how the asymptotic centers of Picard iterative sequences behave. Therefore the main result which we obtained in this direction is the following:

Theorem 4.47. *([P02] Theorem 4.6)*
Let D be a nonempty subset of $\mathbb{H}^\infty$ and $T\colon D \to D$ an asymptotically nonexpansive in the intermediate sense mapping. Moreover, suppose that there is N such that T^N is nonexpansive. Then T has at least one fixed point if and only if there is $x \in D$ for which the Picard iterative sequence $(T^n x)$ is bounded and there is unique $y \in D$ such that

$$d(A(s(x_1, \ldots, x_n)), y) = d(A(s(x_1, \ldots, x_n)), D).$$

Now let us explain some notation used above. For each complete CAT(0) space and each finite set of points $x_1, x_2, \ldots, x_n$ one may find a unique point which minimizes the value of

$$\phi(y) = \sum_{k=1}^{n} \rho^2(y, x_k)$$

(see [28]). Instead of a finite set one may consider any probabilistic measure μ such that

$$\phi(y) := \int_X d^2(x, y)\mu(x) < \infty$$

for some $y \in X$ (so also for any y). In this case there is also unique point $b(\mu)$ which minimizes the value of ϕ. This point is called a barycenter (see [5, Theorem 2.3.1]). However, in paper [P02] we introduced the function

$$\bar{\phi}(y) = \sum_{k=1}^{n} \cosh d(y, x_k)$$

for a finite set of points $x_1, \ldots, x_n \in X$ which turned out to be much more useful in our considerations. We showed that this function also reaches its minimum at precisely one point, which was called a barycenter of the finite set $x_1, x_2, \ldots, x_n$ and was denoted by $s(x_1, x_2, x \ldots, x_n)$. Moreover, if a sequence (x_n) is bounded then the sequence of barycenters is also bounded and so in CAT(0) spaces its asymptotic center $A(s(x_1, \ldots, x_n))$ is well-defined (it is a singleton).

Let us notice two facts. First, the assumptions that our space was the real Hilbert ball equipped with the hyperbolic metric with $H = \ell^2$ may be easily weakened to the case of all spaces with curvature equal to -1 and having the extension property. Moreover, in the theorem we take the asymptotic center of the sequence of barycenters instead of the asymptotic center of a sequence as in previous cases. One may find a number of easy examples showing that bounded sequences in $\mathbb{R}$ have no relation between such points. However, it is still an open question whether the same situation takes place if one considers the Picard iterative sequence of nonexpansive mapping in an infinite dimensional space (with a constant curvature and the extension property). Hence we do not know whether the counterpart of Theorem 4.47 for nonexpansive mappings coincides with the result from Corollary 4.45.

At the end let us highlight that all results presented above can be directly obtained in $\mathbb{R}$–trees which again would be treated as spaces of constant curvature equal to $-\infty$. Moreover, they also hold for strongly continuous semigroups of contractions defined on a space of constant curvature as was shown in Section 6 of [P02]. Here I omit repeating our reasoning for these cases.

5 Other research results (not contained in the monothematic publication cycle)

[P06] B. Piątek, *On the Sincov functional equation*, Demonstr. Math. 38 (2005), 875–881.

[P07] B. Piątek, *On convex and $\star$-concave multifunctions*, Ann. Polon. Math. 86 (2005), 165–170.

[P08] B. Piątek, *Some properties of convex and $\star$-concave multifunctions*, Ann. Acad. Paedagog. Crac. Stud. Math. 5 (2006), 77–86.

[P09] B. Piątek, *Riemann integrability of a nowhere continuous multifunction*, Ann. Acad. Paedagog. Crac. Stud. Math. 7 (2008), 5–13.

[P10] B. Piątek, *Best approximation of coincidence points in metric trees*, Ann. Univ. Mariae Curie-Sklodowska Sect. A 62 (2008), 113–121.

[P11] B. Piątek, *On the continuity of the integrable multifunctions*, Opuscula Math. 29 (2009), 81–88.

[P12] B. Piątek, *On asymptotic cyclic contractions*, Ann. Acad. Paedagog. Crac. Stud. Math. 9 (2010), 79–86.

[P13] R. Espínola, A. Fernández-León and B. Piątek, *Fixed points of single- and set-valued mappings in uniformly convex metric spaces with no metric convexity*, Fixed Point Theory Appl. 2010, art. no. 169837, 16 pp.

[P14] B. Piątek, *On cyclic Meir-Keeler contractions in metric spaces*, Nonlinear Anal. 74 (2011), 35–40.

[P15] B. Piątek, *Halpern iteration in CAT(κ) spaces*, Acta Math. Sin. (Engl. Ser.) 27 (2011), 635–646.

[P16] B. Piątek, *Viscosity iteration in CAT(κ) spaces*, Numer. Funct. Anal. Optim. 34 (2013), 1245–1264.

[P17] R. Espínola and B. Piątek, *Diversities, hyperconvexity and fixed points*, Nonlinear Anal. 95 (2014), 229–245.

[P18] B. Piątek, *On the gluing of hyperconvex metrics and diversities*, Ann. Univ. Paedagog. Crac. Stud. Math. 13 (2014), 65–76.

[P19] G. López-Acedo and B. Piątek, *Characterization of compact geodesic spaces*, J. Math. Anal. Appl. 425 (2015), 748–757.

[P20] B. Piątek, *A survey of the fixed point property in CAT (κ) spaces*, in Monograph on the occasion of 100th birthday anniversary of Zygmunt Zahorski, Ed. by Roman Wituła, Damian Słota, Waldemar Hołubowski. Gliwice: Wydaw. Politechniki Śląskiej, 2015, 355–364.

[P21] G. López-Acedo and B. Piątek, *Some remarks on a characterization of compactness by means of the fixed point property in geodesic spaces*, J. Nonlinear Convex Anal. 17 (2016), 1259–1263.

[P22] M. Adam, B. Piątek, M. Pleszczyński, B. Smoleń and R. Wituła, *Certain inequalities connected with the golden ratio and the Fibonacci numbers*, J. Appl. Math. Comput. Mech. 15 (2016), 5–15.

[P23] M. Adam, B. Piątek, M. Pleszczyński, B. Smoleń and R. Wituła, *On values of the psi function*, J. Appl. Math. Comput. Mech. 16 (2017), 7–18.

[P24] E. Hetmaniok, B. Piątek and R. Wituła, *Binomials transformation formulae for scaled Fibonacci numbers*, Open Math. 15 (2017), 477–485.

[P25] G. López-Acedo, A. Nicolae and B. Piątek, *"Lion-man" and the fixed point property*, under review.

[P26] B. Piątek and A. Samulewicz, *Gluing Busemann spaces*, in Selected problems on experimental mathematics, Ed. by Roman Wituła, Beata Bajorska-Harapińska, Edyta Hetmaniok, Damian Słota, Tomasz Trawiński. Gliwice: Wydaw. Politechniki Śląskiej, 2017, 129–148.

[P27] E. Hetmaniok, B. Piątek, M. Pleszczyński and R. Wituła, *Identities for the inverses of central binomial coefficients*, in Selected problems on experimental mathematics, Ed. by Roman Wituła, Beata Bajorska-Harapińska, Edyta Hetmaniok, Damian Słota, Tomasz Trawiński. Gliwice: Wydaw. Politechniki Śląskiej, 2017, 219–231.

[P28] B. Piątek, On the geometry and fixed point free mappings in hyperbolic spaces, J. Fixed Point Theory Appl., to appear.

Now I am going on to discuss my other papers which are not contained in the main scientific achievement. In more detail, I will focus only on the most essential ones, also devoted to geodesic spaces.

5.1 Approximation of fixed points in spaces with a positive upper bound on the curvature

The papers [P15] and [P16] are devoted to the problem of finding fixed points for a nonexpansive mapping T defined on a complete CAT(κ) space X (not necessary bounded) with positive parameter κ.

It is worth emphasizing that [P15] was the first paper where the author analyzed the problem of approximating fixed point of nonexpansive self-mappings in the case of spaces with curvature bounded above by a positive number where standard methods used for instance in Banach spaces and W-hyperbolic spaces do not work. During the next few years the interest of this topic was increasing, which can be seen if one analyzes the application of results from [P15] in the development of research on effective regularity of approximation due to U. Kohlenbach, L. Leuştean and A. Nicolae (see for instance [45, Theorem 3.1] and [37]).

First, in the paper [P15] I presented the solution of the problem by using the Halpern iterative process (introduced in [26] by B. Halpern) of the form

$$x_{n+1} = t_n x_n + (1 - t_n)Tx_n, \tag{5.1}$$

where the sequence of numbers (t_n) satisfies the following conditions:

(i) $t_n \in (0, 1)$ and $t_n \to 0$;

(ii) $\sum t_n = \infty$;

(iii) $\sum |t_{n+1} - t_n| < \infty$.

The main result of my paper is the following one:

Theorem 5.1. *([P15] Theorem 4.2)*
Let κ be an arbitrary positive number, X a complete CAT(κ) space and $T\colon X \to X$ a nonexpansive mapping such that $Fix(T) \neq \emptyset$. If (t_n) is a sequence satisfying conditions (i)–(iii), then for each $x_0 \in X$ such that $\rho(x_0, Fix(T)) \leq \frac{\pi}{4\sqrt{\kappa}}$, the sequence (x_n), defined by (5.1)*, tends to the nearest fixed point to x_0.*

My reasoning can be sketched as follows. Let q be the projection of a point $x_0 \in X$ onto the set of fixed points of T. Let us notice that we do not consider the case of CAT(0) space, so we cannot apply (iii) of Proposition 4.2 and the existence and uniqueness of q follows from the fact that closed and convex subsets of CAT(κ) spaces with sufficiently small diameter are uniformly convex. Hence for each $t \in (0, 1)$, a mapping $T_t\colon \bar{B}\left(q, \frac{\pi}{4\sqrt{\kappa}}\right) \to$

$\bar{B}\left(q, \frac{\pi}{4\sqrt{\kappa}}\right)$ defined by $T_t(x) = tx_0 + (1-t)T(x)$ is a contraction and the sequence (z_t) of fixed points $z_t = T_t(z_t)$ tends to q for $t \to 0^+$ (see [P15, Theorem 3.5]). At the same time, under the assumptions from Theorem 5.1 it can be proved that the sequence (x_n) satisfies $\rho(x_n, T(x_n)) \to 0$. Using these two facts I showed that Theorem 5.1 is true.

Let us notice that in the proof the main result I introduced some properties of spherical geometry which were applied afterwards by other authors to solve similar problems for various types of iterations (see Lemma 3.2 and 3.3 in [P15]).

The paper [P16] was also devoted to the solution of the approximation of fixed point for nonexpansive mappings in the same spaces but by applying a two-step modification of the following viscosity iteration introduced by A. Moudafi [47]

$$x_{n+1} = t_n f(x_n) + (1-t_n)Tx_n, \tag{5.2}$$

where f is a contraction defined on a subset of X. In the case of CAT(1) spaces I needed to impose stronger assumption of f – we will come back to this discussion in the sequel.

Let us suppose that two sequences of real numbers (b_n) and (t_n) satisfy the following conditions:

(i) $b_n, t_n \in (0,1)$ for $n \in \mathbb{N}$;

(ii) $0 < \liminf\limits_{n\to\infty} b_n \leq \limsup\limits_{n\to\infty} b_n < 1$;

(iii) $\lim\limits_{n\to\infty} t_n = 0$;

(iv) $\sum\limits_{n=1}^{\infty} t_n = \infty$ or equivalently $\prod\limits_{n=1}^{\infty}(1-t_n) = 0$.

Using these sequences in [P16] I proved that

Theorem 5.2. *([P16] Theorem 4.2)*
Let C be a complete CAT(κ) space, $\kappa > 0$, with property N and diameter smaller than $D_\kappa/2$. Moreover, let $T\colon C \to C$ be a nonexpansive mapping with $FixT \neq \emptyset$ and let $f\colon C \to C$ be k-contractive with

$$k < \frac{2\sin^2\frac{M}{2}\cos M}{M^2}. \tag{5.3}$$

Assume further that $\rho(p, f(p)) \leq M/4\sqrt{\kappa}$ for all $p \in FixT$, where $M \in (0, \pi/2)$. Then there is a unique fixed point $q \in FixT$ for which

$$P_{FixT}(f(q)) = q. \tag{5.4}$$

Moreover, for each point $u \in X$ such that

$$\rho(u, q) \leq M/4\sqrt{\kappa}$$

and for each couple of sequences (b_n) and (t_n) satisfying (i)–(iv), the viscosity iterative sequence defined by

$$\begin{aligned} x_1 &= u, \\ y_n &= t_n f(x_n) + (1-t_n)T(x_n), \\ x_{n+1} &= b_n x_n + (1-b_n) y_n \end{aligned} \tag{5.5}$$

converges to q satisfying (5.4).

Note that in order to obtain the main result I needed to suppose additionally that the space X satisfies the so-called property N (the property of nice projections). As far as I know there is no the correct result about the viscosity approximation in CAT(κ) spaces, $\kappa > 0$, without property N.

Definition 5.3. A CAT(κ) space X is said to satisfy *property* N if for any closed D_κ-convex set $C \subset X$ and for each points $x_1, x_2 \in X$ such that $\rho(x_i, C) < \frac{D_\kappa}{2}$, $\rho(x_1, x_2) < \frac{D_\kappa}{2}$ and $P_C(x_1) = P_C(x_2) = P$, the projection $P_C(\alpha x_1 + (1-\alpha)x_2)$, $\alpha \in (0,1)$, is also equal to P.

As a consequence, if property N holds, then the continuity of the metric projection implies that $P_{[a,b]}(\alpha x_1 + (1-\alpha)x_2) \in [P_{[a,b]}(x_1), P_{[a,b]}(x_2)]$ for any metric segment $[a, b]$. Property N has been recently introduced in [17] where the authors claimed that property N was to be very common within the class of CAT(κ) spaces but no such space failing property N was provided (see Question in [17]). So in [P16] I proposed the first example in this sense.

Example 5.4. ([P16] Example 5.1)
Let us consider two flat triangles in $\mathbb{E}^3$

$$A = (-1, 0, 4); B = (1, 0, 4); C = (0, 0, 0)$$

and

$$C = (0, 0, 0); D = (0, 0, 4); E = \left(0, \frac{3\sqrt{7}}{8}, \frac{1}{8}\right).$$

Now let us define a distance in the space $X = \Delta(A, B, C) \cup \Delta(C, D, E)$ as the length of the shortest path in X connecting two points. To check that our space is CAT(0) it suffices to notice that X is a gluing space of two subsets of $\mathbb{E}^2$, so one may apply Reshetnyak's Gluing Theorem.

We will prove that

$$P_{[C,E]}(D) = P_{[C,E]}\left(\frac{1}{2}A + \frac{1}{2}B\right) \notin \left[P_{[C,E]}(A), P_{[C,E]}(B)\right]. \tag{5.6}$$

Clearly, the projection of D is equal to $F = \frac{1}{2}C + \frac{1}{2}E = \left(0, \frac{3\sqrt{7}}{16}, \frac{1}{16}\right)$ because $\angle_F(D,E) = \frac{\pi}{2}$. Now let us calculate

$$d(A,C) = \sqrt{1^2 + 4^4} = \sqrt{\frac{136}{8}}$$

and

$$d(A,F) = \sqrt{\left(\frac{3\sqrt{7}}{16} + 1\right)^2 + \left(\frac{1}{16} - 4\right)^2} = \sqrt{\frac{134 + 3\sqrt{7}}{8}} > \sqrt{\frac{136}{8}}.$$

Because of symmetry $P_{[C,E]}(A) = P_{[C,E]}(B) \neq F$. This completes the proof of (5.6).

The main tool using in the proof of Theorem 5.2 was the function h whose construction is similar to the quadrilateral cosine introduced by I. Berg and I. Nikolaev in [9].

Let us suppose that X is a CAT(1) space and define the function h as

$$h(A,B;C,D) = \frac{\cos\rho(A,C) + \cos\rho(B,D) - \cos\rho(A,D) - \cos\rho(B,C)}{\rho(A,B)\rho(C,D)}$$

for each four points A, B, C, D of X such that

$$\max_{x,y\in\{A,B,C,D\}} d(x,y) < \pi/2 \quad \text{and} \quad A \neq B,\ C \neq D.$$

This function has some nice properties, which allowed me to obtain similar results concerning the boundedness of h to the ones given by Berg and Nikolaev (see [9]) for the quadrilateral cosine.

Lemma 5.5. *([P16] Lemma 3.10) Let X be a CAT(1) space. Then the inequality*

$$|h(A,B;C,D)| \leq 1$$

holds for each four of points A, B, C, D of X such that $A \neq B$, $C \neq D$ and $\max_{x,y\in\{A,B,C,D\}}(x,y) < \frac{\pi}{2}$.

Let us notice that one may also get the counterpart of Theorem 5.2 in the case of CAT(0) spaces:

Theorem 5.6. *(Theorem 4.3 [P16]).*
Let C be a complete CAT(0) space with property N. Let $T\colon C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$ and let $f\colon C \to C$ be k-contraction $k < \frac{1}{2}$. Then there is a unique fixed point $q \in Fix(T)$ for which (5.4) *holds. Moreover, for each point $u \in X$ and for each couple of sequences (b_n) and (t_n) satisfying (i)–(iv), the viscosity iterative sequence defined by* (5.5) *converges to q.*

For CAT(0) spaces no estimations of distances are needed and I only assumed that

$$k < \frac{1}{2}. \tag{5.7}$$

Perhaps it seems a bit artificial that the last result does not hold for all contraction. The reason why (5.7) is imposed has to do with the fact that rather than applying typical methods from linear spaces I use the function h which is strongly related to the spherical geometry.

At the end of our consideration let us come back to the assumption on κ in the case of CAT(κ) spaces. This assumption follows from considering a composition of mappings which in the end needed to be a contraction. One of the mappings was the projection onto the set of fixed points. In contrast to the situation in CAT(0) spaces, the projection onto a closed convex and bounded subset C of a complete CAT(1) space is not always nonexpansive. More precisely, this projection is a Lipschitzian mapping with a constant L depending on the diameter of C in the following way

$$L = \frac{M}{2\arcsin(\sin(M/2)\cos(M))},$$

where $M < \pi/2$ is a diameter of C. The precise value of L was obtained by me and at the request of G. López-Acedo was included in [2, Proposition 3.4.].

5.2 Fixed point property for continuous mapping

Simultaneously to the problem of existence of fixed points for nonexpansive mappings I worked on the similar problem for continuous mappings defined on geodesic spaces. Let us begin with the very well known result due to Klee from 1955:

Theorem 5.7. *([36] Theorem 2.3)*
For a convex set K of a locally convex metrizable topological linear space E, the following assertions are equivalent:

(α) K is compact;

(β) K has the fixed points property;

(γ) no relatively closed subset of K is a topological ray.

Let us recall that the image of isomorphic embadding $c\colon [0,\infty) \to X$ is called the topological ray and this ray can be defined in any topological space.

So the natural question which can be raised here is how far this result is related to the geometry of the space. In other words, one may raise the question whether its counterpart still holds in nonlinear spaces. Our results obtained in cooperation with Genaro López-Acedo were motivated by two papers: the first one due to D. Ariza-Ruiz, C. Li, G. López-Acedo [4] and the second one due to C. P. Niculescu and I. Roventa [49]. The results contained in these papers can be summarized in the following form:

Theorem 5.8. *(see [4, Theorem 16] and [49, Theorem 1.3])*
Let (X,d) be a uniquely geodesic space such that it satisfies property (P) and all balls are convex. Let K be a nonempty closed convex subset of (X,d). Then, any continuous mapping $T\colon K \to K$ with a compact range $T(K)$ has at least one fixed point in K.

The property (P) in this theorem has a geometrical meaning and says that

$$\limsup_{\varepsilon\searrow 0} d((1-t)x+ty,(1-t)x+tz\colon t\in[0,1],x,y,z\in X,d(y,z)\le\varepsilon)=0.$$

All spaces mentioned before satisfy the property (P). Instead of compactness of the range of T one may consider the compactness of K, so this leads to a natural question whether in the settings from Theorem 5.8 or at least in CAT(κ) spaces (with any $\kappa\in\mathbb{R}$) the fixed point property of continuous mappings is equivalent to the compactness of the domain K. A not too complicated example of $\mathbb{R}$–trees shows that in general this is not true. Since $\mathbb{R}$–trees may be treated as spaces of curvature equal to $-\infty$ the next question which can be raised is whether imposing a lower bound on the curvature changes this situation. A positive result for this question was given in [P19] in the following form:

Theorem 5.9. *([P19] Theorem 10)*
Let X be a complete uniquely geodesic space with curvature bounded below which satisfies property (P) and has convex balls. Then the following are equivalent:

a) X is compact;

b) X does not contain any closed topological ray;

c) X has the fixed point property for continuous mappings.

In the same paper we gave an example of a noncompact CAT(0) space for which the curvature was not bounded below and which satisfies the fixed point property for continuous mappings. Moreover, there was no subset of this space isometric to a tripoid.

In the paper [P21] written after fruitful discussions at the 11th International Conference of Fixed Point Theory and Applications we noticed that in the case of locally compact geodesic spaces the uniqueness of geodesics is sufficient to obtain property (P) and all compact and locally convex subsets has the fixed point property (see [P21, Corollary 2.10]). Finally in [P25] we gave the final geometrical characterization of the fixed point property for continuous mappings in locally compact geodesic spaces:

Theorem 5.10. *([P25] Theorem 4)*
Let (X, d) be a complete, locally compact, uniquely geodesic space and $A \subset X$ nonempty, closed and convex. Then A has the fixed point property for continuous mappings if and only if A is compact.

5.3 Diversity and hyperconvexity of spaces

In the papers [P17] and [P18] we focused on the notion of diversity and the generalization of hyperconvexity for diversities. Let us begin with some notions which will be used in the sequel of this section.

Definition 5.11. ([19] Definition 2.5)
The metric space (X, d) is said to be hyperconvex if for each collection of points $\{x_\alpha\}_{\alpha \in I}$ from X and positive numbers $\{r_\alpha\}_{\alpha \in I}$ such that $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ the following holds true

$$\bigcap_{\alpha \in I} \bar{B}(x_\alpha, r_\alpha) \neq \emptyset. \tag{5.8}$$

From the definition each hyperconvex metric space is geodesic but not necessary uniquely geodesic. Furthermore, it is known that each hyperconvex and uniquely geodesic space must be a metric tree (see [29]). Moreover, for each metric space (X, d) one may consider the tight span of X being the smallest hyperconvex space containing an isometric copy of X. This concept was introduced by N. Aronszajn and P. Panitchpakdi in the fifties (see [19]).

Motivated by problems in phylogenetic and ecological diversity, D. Bryant and P. F. Tupper in [12] introduced the mathematical concept of diversity, i.e., space in which metric is replaced by the diversity of finite sets and showed how the notion of tight span worked in this case.

Definition 5.12. ([12] Definition 2.8)
Let X be a set and denote by $\langle X\rangle$ the set of its finite subsets, then a diversity is a pair (X, δ), where $\delta\colon \langle X\rangle \to \mathbb{R}$ such that

1. $\delta(A) \geq 0$ and $\delta(A) = 0$ iff $|A| = 1$, where $|A|$ stands for the cardinality of A.
2. If $B \neq \emptyset$ then $\delta(A \cup C) \leq \delta(A \cup B) + \delta(B \cup C)$.

It is obvious that each metric space (X, ρ) can be interpreted as a diversity with

$$\delta(A) = \operatorname{diam}(A), \qquad A \in \langle X\rangle \tag{5.9}$$

and at the same time diversity is a metric space with

$$d(x, y) = \delta(\{x, y\}). \tag{5.10}$$

Also the concept of tight span can be generalized for diversities in the following way

Definition 5.13. ([12] Definition 2.12)
A diversity (A, δ) is said to be hyperconvex if for all $r\colon \langle X\rangle \to \mathbb{R}$ such that

$$\delta\left(\bigcup_{A\in\mathcal{A}} A\right) \leq \sum_{A\in\mathcal{A}} r(A),$$

for all $\mathcal{A} \subset \langle X\rangle$ finite, with $r(\emptyset) = 0$, there is $z \in X$ such that $\delta(\{z\} \cup Y) \leq r(Y)$ for all finite $Y \subset X$.

In the paper [P17] written in cooperation with Rafa Espínola we considered two problems. The first one was to study the relation between hyperconvexity of diversities and hyperconvexity of metric spaces. First we gave an example (see Example 3.1 in [P17]) showing that if the diversity (X, δ) is hyperconvex then the metric space equipped with the induced metric given by (5.10) does not have to be hyperconvex. However we proved that:

Proposition 5.14. *([P17] Proposition 3.3)*
Let (X,δ) be a hyperconvex diversity and (X,d) its induced metric space. If for any $A=\{x_1,\dots,x_n\}\in\langle X\rangle$ we have that

$$(|A|-1)\cdot\delta(A)\leq\sum_{1\leq i<j\leq n}d(x_i,x_j),$$

then (X,d) is a hyperconvex metric space.

The next problem we focused on in [P17] was the fixed point property for nonexpansive mappings defined on bounded spaces. Let us notice that first we considered mappings with a domain bounded with respect to the induced metric. However, as shown in [P17, Theorem 3.11] we may find hyperconvex diversities with bounded induced metric and nonexpansive mappings defined on them in such a way that the mappings are fixed point free. Rather then bounding the induced metric, a much more useful condition proved to be the following one: we say that the diversity (X,δ) is bounded if there is a positive M such that for each $A\in\langle X\rangle$ we have $\delta(A)\leq M$. Then one may show that:

Theorem 5.15. *([P17] Theorem 4.2)*
Let (X,δ) be a hyperconvex and bounded diversity with induced metric space (X,d), and $T\colon(X,d)\to(X,d)$ a nonexpansive mapping. Then T has a fixed point in X .

This theorem extends the result due to J. B. Baillon (see [6, Theorem 5]) about the existence of fixed points for nonexpansive mappings defined on bounded hyperconvex metric spaces, because each hyperconvex metric space is hyperconvex with respect to the diameter diversity defined by (5.9) (see Lemma 4.2 in [12]).

While working on the paper we came across the problem of building new diversities via gluing others. I focused on this topic in the paper [P18] where I studied when the gluing of diversities is again a diversity with δ defined by:

Proposition 5.16. *([P18] Proposition 2.1)*
Let X and Y be two sets such that $X\cap Y=\{\theta\}$ and (X,δ_X) and (Y,δ_Y) are diversities. Then $(X\cup Y,\delta)$ is a diversity with

$$\begin{aligned}\delta(A)\ &=\ \delta_X((A\cap X)\cup\{\theta\})+\delta_Y((A\cap Y)\cup\{\theta\}),\\ &\qquad A\cap(X\setminus\{\theta\})\neq\emptyset,\ A\cap(Y\setminus\{\theta\})\neq\emptyset\end{aligned}$$

and $\delta(A)=\delta_X(A)$ (or $\delta(A)=\delta_Y(A)$) for $A\subset X$ (or $A\subset Y$), respectively.

Then we get:

Theorem 5.17. *([P18] Theorem 2.2)*
Let (X, δ) and (Y, δ) be two hyperconvex diversities with $X \cap Y = \{\theta\}$. Then $(X \cup Y, \delta)$ with the function δ defined as in Proposition 5.16 is hyperconvex.

Recently, research on the problem of gluing hyperconvex diversities and hyperconvex metric spaces was continued by, among others, B. Miesch and M. Borkowski.

5.4 The other papers

The papers [P06] – [P09] and [P11] were related to the topic of my PhD thesis and were devoted to set-valued functions. More precisely, I analyzed the assumptions under which multifunction with values which are closed convex and bounded subsets of Banach spaces, are integrable with respect to the Riemann definition. Moreover, I studied relations between this type of integrability, the existence of Hukuhara derivatives and integrability in the sense of Aumann. The papers [P06] – [P08] were published before my PhD thesis defense.

The paper [P10] included my first results devoted to the topic of geodesic spaces. I studied the existence of fixed points for combinations of multifunctions defined on $\mathbb{R}$–trees and I generalized Theorem 4.9 due to W. A. Kirk.

The next paper – [P13] – written in cooperation with Rafa Espínola and Aurora Fernández-León was devoted to the existence of fixed points for several types of asymptotically nonexpansive mappings defined on uniformly convex geodesic spaces.

The papers [P12] and [P14] were devoted to the cyclic contractions. Let us suppose that A, B are two subsets of a metric space and a mapping $T\colon A \cup B \to A \cup B$ such that $T(A) \subset B$ and $T(B) \subset A$. We consider whether there exists a pair $(x_0, y_0) \in A \times B$ for which $d(x_0, y_0) = \inf\{d(x, y) \,:\, x \in A, y \in B\}$. In [P12] I focused on asymptotic cyclic contractions and in [P14] – the cyclic Meir-Keeler contractions. I gave, among others, the example of a pair of sets which satisfy the so-called WUC property but not the UC one, which answer the question from [18]. Moreover, I showed that in contrast to the case of cyclic contractions (see [18]) there exists a pair of sets which satisfies the WUC property and the cyclic Meir-Keeler contraction T such that there does not exist a pair of best proximation (compare to [61]).

The paper [P26] was devoted to the gluing of Busemann convex spaces. More precisely, in cooperation with Alicja Samulewicz I studied under which additional assumptions the gluing of Busemann convex spaces is the space from the same class. We have already mentioned these results at pages 9–10, where we discussed the gluing of geodesic spaces.

The paper [P28] was devoted to the metrization problem of the geodesic boundary of Busemann convex space equipped with the cone topology.

The paper [P20] was the overview paper when I focused on the latest results on the fixed point property in CAT(0) spaces.

Simultaneously I was a member of the research group led by Roman Wituła, where we studied properties of real sequences ans series. In particular we were interested in the sequences of Fibonacci and Lucas numbers, and also central binomial coefficients. Our results were published, among others, in the following papers: [P22] – [P24] and [P27].

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