Limit theorems for motions in random fields

Summary

Motivation

My thesis refers to turbulent diffusion. It is a transport of mass, heat, or momentum in liquid. This flow is very complicated and not laminar. The complexity of a flow is characterized by the Reynolds number *Re*. The turbulent flow of the liquid for sufficiently large values of *Re* is characterized by the fact that in each point of stream we have very irregular and disordered changing of speed in time. A typical value of *Re*, for which we can observe fully developed turbulence is approx. 2000 [17]. For example, the Reynolds number for a flow of a particle in viscous liquid such as honey or glycerine is less that 1. Turbulence occurs much more rapidly than molecular diffusion and is therefore extremely important for problems concerning mixing and transport. This issue is also cold Arnold's diffusion [1]. The physical background of this problem can be found in the Frisch monograph [7]. According to Fick's law ([7]) in turbulence transport the square mean displacement is proportional to the time. However in turbulent flow may occurs some anomalies in asymptotic behavior. For example we can observe the superdiffusive behavior (the square mean displacement grows faster than time) or the subdiffusive. In [8],[2] authors show that then there is not effective diffusivity and the asymptotic behavior is not given by simple diffusion equation.

In classical turbulent case one of the most interesting issues is a study of the asymptotic behavior, with the assumption that the molecular diffusivity vanishes. The concept of a turbulent diffusion was introduced by G.I. Taylor in early 1920s in [22]. R. Kraichnan [15] conjectured that the diffusion approximation holds even for zero mean, divergence-free velocity field $V(x)$ that are independent of t.

Until now most, of the considered models deal with stationary fields. It is because we have to use ergodic theorem. We want to deal with a mathematically more difficult case. Namely, we will consider differential equation with random but non stationary coefficients.

Passive Tracer Model

One of the simplest formulations of the problem of a turbulent diffusion and the problem of a turbulent transport is a *passive tracer model*. This model is described by the following formula

$$\begin{cases} \dfrac{dX(t)}{dt} = \mathbf{V}\left(t, X(t)\right), \\ X(0) = 0, \end{cases}$$

where $V(t,x)$ is some random field. It is a classical model of statistic hydrodynamics and it is described in details in monograph [18]. The physical interpretation of this model may be the observation of a particle of a pollution moving in a turbulent flow. This particle is called a tracer. The trajectory of the tracer is modeled by the process $X(t)$. The equation describes microscopic dynamics, but observations are done in a macroscopic scale. So we need to apply some rescaling in order to have the situation that the macroscopic observation shows the microscopic changes. The passive tracer model after diffusive rescaling is as follows

$$\begin{cases} \dfrac{dX^{(\varepsilon)}(t)}{dt} = \dfrac{1}{\varepsilon} V\left(\dfrac{t}{\varepsilon^2}, \dfrac{X^{(\varepsilon)}(t)}{\varepsilon}\right) \\ X^{(\varepsilon)}(0) = 0. \end{cases} \tag{1}$$

The above equation was studied mostly in the case when, the field V is divergent-free and stationary ([4],[9],[10],[13],[6]).

One of the basic questions in studying asymptotic behavior of the tracer is the question about the law of large numbers (LLN) and the central limit theorem (CLT). The law of large numbers states that there exists a deterministic vector $v_* \in \mathbb{R}^d$ (called *Stoke's drift*), such that $X(t)/t \to v_*$ a.e. If it is possible to show LLN, then we can ask about CLT, i.e. does $(X(t) - v_* t)/\sqrt{t}$ converge in distribution when $t \to +\infty$ to normal vector $N(0,\kappa)$? The covariance matrix $\kappa = [\kappa_{i,j}],\ i,j = 1,\dots d$ is called the *turbulent diffusivity* of the tracer.

Some of the results in the literature

With the assumption that the vector field $\{V(t,x), x \in \mathbb{R}^d\}$ is stationary and Markovian, CLT was proved in [4],[6]. Another important CLT was obtained in 1997 by [13] (see also [5]). Authors of this result deals with zero mean, stationary, divergent-free, Gaussian and T-dependent field $V(t,x)$. T-dependence means that there exists some $T > 0$, such that covariance matrix satisfies $|\mathbb{R}(t,x)| = 0,\ |t| > T$.

Obtained results

On positivity of the variance of a tracer moving in a divergence-free Gaussian random field

In Chapter 3 we consider the field V, which is T-dependent, and satisfy the assumptions from [13]. Recall that in this paper authors established the convergence in law of $X(t)/\sqrt{t}$ to normal the vector $N(0,\kappa)$. The question whether κ does not degenerate and becomes a null matrix has not been adressed in [13]. The fact that $\kappa = 0$ would indicate a subdiffusive behavior of the tracer. In this Chapter we formulate a sufficient condition for $\kappa \neq 0$.

CLT for nonmarkovian fields

Recall that CLT for the passive tracer model was proved for the field which are T-dependant and in case of stationary, markovian vector field $\{V(t,x), (t,x) \in \mathbb{R}^{1+d}\}$ which has the spectral gap property ([6, 14, 4]). In the gaussian case the covariance matrix for this type of fields is following

$$R_{pq}(t,x) := \mathbb{E}\big[V_p(t,x)V_q(0,0)\big] = \int_{\mathbb{R}^d} e^{ix\cdot\xi-\gamma(\xi)|t|}\hat{R}_{pq}(d\xi), \tag{2}$$

where exists γ_0 such that $\gamma(\cdot) \geq \gamma_0 > 0$ and $\hat{R}(d\xi) = [\hat{R}_{pq}(d\xi)]$, $p,q = 1,\dots,d$, is a Borel measure with values in the space of $d \times d$ hermitian, nonnegative defined matrices.

Based on [13, 6] one may suppose that, CLT can be proved for a traser moving in filed, which a covariance satisfying a fast decay condition for example when exists $a, C > 0$ such that

$$|R_{pq}(t,x)| \leq Ce^{-a|t|},\ p,q = 1,\dots,d,\ \text{for all } (t,x) \in \mathbb{R}^{1+d}.$$

One step in this direction is a result from Chapter 4. We consider there the covariance in which term $\exp\{-\gamma(\xi)|t|\}$ from (2) is replaced by

$$h(|t|,\xi) := \int_{\mathbb{R}} e^{-\gamma(\xi,\lambda)|t|}\mu(d\lambda),$$

where μ is some nonnegative, finite Borel measure on $\mathbb{R}$ and $\gamma : \mathbb{R}^{d+1} \to \mathbb{R}$ is nonnegative Borel measurable function. Therefore we consider the field $V(\cdot)$ with the following covariance matrix

$$R_{pq}(t,x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi}h(|t|,\xi)\hat{R}_{p,q}(d\xi), \qquad p,q = 1,\dots,d.$$

Markovian fields corresponds to the case $\mu(d\lambda) = \delta_0(d\lambda)$. Spectral gap property comes from the assumption $\gamma(\xi,\lambda) \geq \gamma_0 > 0$.

Locally Stationary Fields

In Chapter 5 we move away from the stationarity assumption. A natural generalization of this assumption is a concept of *local stationarity.* We want to study the asymptotic behavior of a tracer moving in a divergence-free, locally stationary vector field. In the homogenization theory the concept of *local stationarity* occurs in [19], [21]. The passive tracer model in a locally stationary field can be described by the following equation

$$\begin{cases} \dfrac{dX^{\varepsilon}(t)}{dt} = \dfrac{1}{\varepsilon}\mathbf{W}\left(\dfrac{t}{\varepsilon^2}, \dfrac{X^{\varepsilon}(t)}{\varepsilon}, X^{\varepsilon}(t)\right), \\ X^{\varepsilon}(0) = 0. \end{cases} \tag{3}$$

We introduce here the third argument of the field. It is the local stationarity parameter, i.e. for fixed $y \in \mathbb{R}^d$ the field $\mathbf{W}(t, x, \varepsilon y)$ is stationary and ergodic. We are interested in a situation when $\varepsilon \ll 1$. Note that this argument changes slowly. Namely, it changes in the macroscopic scale, while the microscopic time and space variables are scaled diffusively. We will study the behavior of the tracer $X^{(\varepsilon)}(t)$, when $\varepsilon \to 0$. The above model of the field is too general for us, to say something about the asymptotic behavior of the tracer. To do so, we will study fields constructed in Chapter 5.

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