Summary
Ph. D. thesis

Multidimensional singular stochastic control problems on a finite time horizon

Author: Marcin Boryc
Supervisor: Łukasz Kruk

Singular stochastic control is a class of problems in which one is allowed to change the drift of a Markov process (usually a diffusion) at a price proportional to the variation of the control used. Admissible controls do not have to be absolutely continuous with respect to the Lebesgue measure and they may have jumps. This setup is natural for many problems of practical interest, including portfolio selection in finance, control of queueing networks and spacecraft control.

One-dimensional singular stochastic control problems are well understood by now. In this case, if the running cost is convex, the optimal control makes the underlying process a diffusion reflected at the boundary of the so-called *nonaction region* $\mathcal{D}$. In the case of a diffusion with time-independent coefficients and discounted cost on the infinite time horizon, $\mathcal{D}$ is just an interval and the value function enjoys C^2 regularity (*smooth fit*). Both C^2-regularity of the value function and the characterization of the optimally controled process have been extended to the case of singular control for the two-dimensional Brownian motion [7]. In $n \geq 3$ dimensions only partial results are known. For example, for optimal control of the Brownian motion on the infinite time horizon, regularity of the boundary of $\mathcal{D}$ away from some "corner points" was shown in [8] and a characterization of the optimal control as a solution of the corresponding modified Skorokhod problem was given in [4].

In this thesis we consider an n-dimensional singular stochastic control problem on a finite time horizon in which state is governed by a linear stochastic differential equation with time-dependent coefficients, the running cost is convex and controls may act in any direction. More precisely, let $T > 0$ be a fixed number representing our time horizon and let $W = (W_t, t \geq 0)$ be a standard n-dimensional Brownian motion relative to a filtration $(\mathcal{F}_t, t \geq 0)$ satisfying the usual conditions, defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Denote by $\mathcal{V}$ the set of all admissible controls v which are left-continuous, adapted to the filtration $(\mathcal{F}_t, t \in [0,T])$ random processes acting from $[0,T]$ into $\mathbb{R}^n$, with P-almost surely bounded variation and such that $v_0 = 0$ P-almost surely. As it is customary in singular stochastic control theory, for $v \in \mathcal{V}$ we write $v_t = \int_0^t \gamma_s d\xi_s$, where $|\gamma_t| = 1$ for every $t \in [0,T]$ and ξ is nondecreasing and left-continuous.

Consider the state process described by the stochastic integral equation

$$y_{xt}(s) = x + \int_t^s \Big(a(r)y_{xt}(r) + b(r)\Big)dr + \int_t^s \sigma(r)dW_{r-t} + v(s-t), \qquad s \in [t,T],$$

where $t \in [0,T]$ is an initial time, $x \in \mathbb{R}^n$ is an initial position, $b : [0,T] \to \mathbb{R}^n$ and $a, \sigma : [0,T] \to \mathbb{M}^{n\times n}$ stand for the drift and the covariance terms. With each control $v \in \mathcal{V}$, we associate a cost given by the payoff functional

$$J_{xt}(v) = \mathbb{E}\left\{\int_t^T f(y_{xt}(s), s)e^{-\int_t^s \alpha(r)dr}ds + \int_t^T c(s)e^{-\int_t^s \alpha(r)dr}d\xi(s-t)\right\},$$

where f, α and c are respectively the running cost, the discount factor and the instantaneous cost per unit of "fuel". We assume that $f(x,t)$ is convex in x.

 03.11.2015 Marcin Boryc

Our purpose is to characterize the optimal cost, the so called value function

$$u(x,t) = \inf\{J_{xt}(v) : v \in \mathcal{V}\}.$$

If this infimum is attained for some $v^* \in \mathcal{V}$, we say that v^* is an optimal policy (for given t and x).

In the first part of the dissertation, under suitable assumptions, we prove estimates for the value function. These estimates imply that the value function has locally bounded generalized derivatives of the second order with respect to the space variable and of the first order with respect to the time variable. Consequently, the spatial gradient Du of the value function is Hölder continuous in both variables. These properties are needed to consider the value function as a solution of the corresponding fully nonlinear parabolic Hamilton-Jacobi-Bellman equation

$$\max\Big\{Au(x,t) - f(x,t)\ ,\ |Du(x,t)| - c(t)\Big\} = 0$$

in some generalized sense. Here A is a parabolic partial differential operator given by

$$Au(x,t) = \frac{-\partial u(x,t)}{\partial t} - \frac{1}{2}\beta(t) \circ D^2 u(x,t) - \Big(a(t)x + b(t)\Big) \circ Du(x,t) + \alpha(t)u(x,t),$$

where $\circ$ denotes the scalar product of vectors or matrices respectively and $\beta(t) = \sigma(t)\sigma^T(t)$. We also prove existence and uniqueness of an optimal control v^*. The above results have already been published in [1].

The second part of the thesis is devoted to a characterization of the optimal policy for an n-dimensional Brownian motion in the parabolic case as a unique solution to the modified Skorokhod problem for the nonaction region $\mathcal{D} = \{(x,t) \in \mathbb{R}^n \times [0,T] : |Du(x,t)| < 1\}$, having time dependent (moving) boundary. More precisely, we define a vector field $\Gamma_t(\cdot) = -Du(\cdot,t)$. Let $(x_0, 0) \in \overline{\mathcal{D}}$. We say that a process $v \in \mathcal{V}$, $v_t = \int_0^t \gamma_s d\xi_s$, is a solution to the modified Skorokhod problem for a Brownian motion $\sqrt{2}W_t$ in $\overline{\mathcal{D}}$ starting at x_0 with reflection direction Γ if

(a) the process $X_t = x_0 + \sqrt{2}W_t + v_t$ satisfies $(X_t, t) \in \overline{\mathcal{D}}$ for $t \in [0,T]$ a.s.,

(b) for every $t \in [0,T]$, $\xi_t = \int_0^t \mathbb{I}[(X_s, s) \in \partial\mathcal{D},\ \gamma_s = \Gamma_s(X_s)]d\xi_s$,

(c) with probability 1, for each $t \in [0,T]$, a possible jump of the process X at time t occurs on some interval $I \subset \mathbb{R}^n$ parallel to the vector field Γ_t on I (i.e. for all $x \in I$ $\Gamma_t(x)$ is parallel to I) and such that $I \times \{t\} \subseteq \partial^*\mathcal{D} \cap (\mathbb{R}^n \times \{t\})$. If X_t encounters such an interval I, it instantaneously jumps to its endpoint in the direction Γ_t on I.

Here $\partial^*\mathcal{D}$ denotes "the lateral boundary" of $\mathcal{D}$ - it is the parabolic boundary of $\mathcal{D}$ without the "bottom".

The main theorem of this thesis states that for every initial position $x_0 \in \mathbb{R}^n$ such that $(x_0, 0) \in \overline{\mathcal{D}}$, the optimal policy v^* for our singular stochastic control problem is a solution to the modified Skorokhod problem for the Brownian motion $\sqrt{2}W_t$ in $\overline{\mathcal{D}}$ starting from x_0 at time 0 with reflection direction Γ. This result is an analog of the main theorem from [4]. It resolves a long-standing open problem on the structure of the optimal control in the case under consideration.

The Skorokhod problem for domains with time-dependent boundaries has been considered, inter alia, in [2, 3, 5, 6]. In these papers, restrictive assumptions on regularity of the boundary of the domain and the directions of reflection are necessary for existence of a solution to the problem. In our case, explicit regularity results for $\partial\mathcal{D}$ are not available and the assumptions on $\mathcal{D}$ are hidden in the very nature of our stochastic control problem. Therefore, we use a direct

03.11.2015 Marcin Borys

probabilistic, control theoretic approach, similar to the one applied in [4]. Our problem, however, is notably more difficult than the elliptic one, considered in [4], because time-dependence in the value function and the nonaction region creates serious problems, which have to be overcome in our analysis.

References

[1] M. Boryc and Ł. Kruk, *A multidimensional singular stochastic control problem on a finite time horizon*, Annales UMCS, VOL. LXIX, No. 1, pp. 23-57, 2015.

[2] K. Burdzy, Z.-Q. Chen, J. Sylvester, *The heat equation and reflected Brownian motion in time-dependent domains*, The Annals of Probability, Vol. 32, No. 1B, pp. 775-804, 2004.

[3] K. Burdzy, W. Kang, K. Ramanan, *The Skorokhod problem in a time-dependent interval*, Stochastic Processes and their Applications, Vol. 119, Issue 2, pp. 428-452, 2009.

[4] Ł. Kruk, *Optimal policies for n-dimensional singular stochastic control problems, Part I: The Skorokhod problem*, SIAM J. Control Optim., 38, No. 5, pp. 1603-1622, 2000.

[5] K. Nyström, T. Önskog, *The Skorokhod oblique reflection problem in time-dependent domains*, The Annals of Probability, Vol. 38, No. 6, pp. 2170-2223, 2010.

[6] M. Słaby, *Explicit solutions of the extended Skorokhod problems in time-dependent bounded regions with orthogonal reflection fields*, Probability and Mathematical Statistics, Vol. 33, Fasc. 1, pp. 121-148, 2013.

[7] H. M. Soner and S.E. Shreve, *Regularity of the value function for a two-dimensional singular stochastic control problem*, SIAM J. Control Optim., 27, pp. 876-907, 1989.

[8] S. A. Williams, P. L. Chow and J. L. Menaldi, *Regularity of the free boundary in singular stochastic control*, Journal of Differential Equations 111, pp. 175-201, 1994.

03. 11. 2015 Marcin Boryc